Lecture 7

Some Generic transformations of quantum states & their effect on the mean vector & covariance matrix:

- Suppose that we would like to shift the mean vector of an n-mode q. state by a vector $\bar{r} \in \mathbb{R}^{2n}$.

- Define the unitary displacement operator $\hat{D}_{\bar{r}}$ as

$$\hat{D}_{\bar{r}} = e^{i \bar{r}^T \Omega \hat{r}}$$

vector of quadrature operators

$$\Omega = \bigoplus_{i=1}^{n} \Omega_1$$

$$= I_n \otimes \Omega_1$$

Note that

$$\bar{r}^T \Lambda \hat{r} = \sum_{j=1}^{2n} \bar{r}_j \Lambda_{jk} \hat{r}_k$$

$$= \sum_{j=1}^{n} \left(\bar{x}_j \hat{p}_j - \bar{p}_j \hat{x}_j \right)$$

Due to this structure, we have that

$$\hat{D}_{\bar{r}} = \hat{D}_{\bar{r}_1} \otimes \hat{D}_{\bar{r}_2} \otimes \cdots \otimes \hat{D}_{\bar{r}_n}$$

$$\text{w/ } \bar{r}_j = \begin{bmatrix} \bar{x}_j \\ \bar{p}_j \end{bmatrix}$$

So a displacement of an n-mode state is a tensor product of single-mode displacements

can think of

$\bar{r}^T \Lambda \hat{r}$ as a Hamiltonian

& note from the above that

$$\left(\bar{r}^T \Lambda \hat{r} \right)^\dagger = \bar{r}^T \Lambda \hat{r}$$

then since this is a Hamiltonian,

It follows that $e^{i \bar{r} \Lambda \hat{r}}$ is unitary

2a

Also, observe that

$$\hat{D}_{\bar{r}}^{\dagger} = \left(e^{i\bar{r}^{T}\Omega\hat{r}}\right)^{\dagger}$$

$$= e^{-i\bar{r}^{T}\Omega\hat{r}}$$

$$= \hat{D}_{-\bar{r}}$$

So we can invert the displacement by displacing in the opposite way...

2b

What are the commutation relations of two different displacement operators $\hat{D}_{r_1}$ & $\hat{D}_{r_2}$

for $r_1, r_2 \in \mathbb{R}^{2n}$?

We prove that

$$\boxed{\hat{D}_{r_1+r_2} = \hat{D}_{r_1} \hat{D}_{r_2} e^{i r_1^T \Lambda r_2 / 2}}$$

Starting point is the formula

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \dots$$

Now suppose that $[X,Y]$ commutes w/ X & Y

then for real $s \in \mathbb{R}$,
we have that

$$e^{sX} Y e^{-sX} = Y + s[X,Y]$$

Define $g(s) = e^{sX} e^{sY}$ 2c

Differentiate w/ respect to s to get

$$\frac{dg(s)}{ds} = \frac{d}{ds}\left(e^{sX} e^{sY}\right)$$

$$= X e^{sX} e^{sY} + e^{sX} Y e^{sY}$$

$$= X g(s) + e^{sX} Y e^{-sX} e^{sX} e^{sY}$$

$$= \left(X + e^{sX} Y e^{-sX}\right) g(s)$$

$$= \left(X + Y + s[X,Y]\right) g(s)$$

Solution to this diff. eq. is

$$e^{s(X+Y) + \frac{s^2}{2}[X,Y]}$$

$$\Rightarrow g(s) = e^{sX} e^{sY} = e^{s(X+Y) + \frac{s^2}{2}[X,Y]}$$

$\forall s \in \mathbb{R}$ s.t. $[X,Y]$ commutes w/ X & Y

set $s = 1$

$$\Rightarrow e^{X} e^{Y} = e^{X+Y+\frac{1}{2}[X,Y]}$$

2d

Now use this to prove claim

$$\hat{D}_{r_1} \hat{D}_{r_2} = e^{\overbrace{i r_1^T \Lambda \hat{r}}^{}}_{X} \, e^{\overbrace{i r_2^T \Lambda \hat{r}}^{}}_{Y}$$

$$[X, Y] = [i r_1^T \Lambda \hat{r}, i r_2^T \Lambda \hat{r}]$$

$$= -[r_1^T \Lambda \hat{r}, r_2^T \Lambda \hat{r}]$$

$$= -\left[\sum_{jk} r_{1,j} \Lambda_{jk} \hat{r}_k , \sum_{lm} r_{2l} \Lambda_{lm} \hat{r}_m\right]$$

$$= -\sum_{jklm} r_{1,j} r_{2l} \Lambda_{jk} \Lambda_{lm} [\hat{r}_k, \hat{r}_m]$$

$$= -\sum_{jklm} r_{1,j} r_{2,l} \Lambda_{jk} \Lambda_{lm} i \Lambda_{km}$$

$$= -i \sum_{jklm} r_{1,j} \Lambda_{jk} \Lambda_{km} \Lambda_{lm} r_{2,l}$$

Λ antisymmetric

$$= i \sum_{jklm} r_{1,j} \Lambda_{jk} \Lambda_{km} \Lambda_{ml} r_{2,l}$$

$$= i r_1^T \Lambda \Lambda \Lambda r_2$$

$$= -i r_1^T \Lambda r_2$$ ← scalar, commutes w/ X & Y

2e

$$\Rightarrow \quad e^{i r_1^T \Lambda \hat{r}} e^{i r_2^T \Lambda \hat{r}}$$

$$= e^{i r_1^T \Lambda \hat{r} + i r_2^T \Lambda \hat{r} - \frac{i}{2} r_1^T \Lambda r_2}$$

$$= e^{i (r_1 + r_2)^T \Lambda \hat{r}} e^{-\frac{i}{2} r_1^T \Lambda r_2}$$

$$= \hat{D}_{r_1 + r_2} e^{-\frac{i}{2} r_1^T \Lambda r_2}$$

$$\Rightarrow \boxed{\hat{D}_{r_1 + r_2} = \hat{D}_{r_1} \hat{D}_{r_2} e^{\frac{i}{2} r_1^T \Lambda r_2}}$$

Can apply twice to get

$$\hat{D}_{r_1} \hat{D}_{r_2} e^{\frac{i}{2} r_1^T \Lambda r_2} = \hat{D}_{r_1 + r_2}$$

$$= \hat{D}_{r_2} \hat{D}_{r_1} e^{\frac{i}{2} r_2^T \Lambda r_1}$$

$$= \hat{D}_{r_2} \hat{D}_{r_1} e^{\frac{i}{2} r_1^T \Lambda^T r_2}$$

$$= \hat{D}_{r_2} \hat{D}_{r_1} e^{-\frac{i}{2} r_1^T \Lambda r_2}$$

$$\Rightarrow \boxed{\hat{D}_{r_1} \hat{D}_{r_2} = \hat{D}_{r_2} \hat{D}_{r_1} e^{-i r_1^T \Lambda r_2}}$$

The connection between the traditional single-mode displacement operator & this convention is as follows:

for $\alpha \in \mathbb{C}$ & $\alpha = \alpha_R + i\alpha_I$

$$D(\alpha) \equiv \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right)$$

$$= \exp\left(\left[\alpha_R + i\alpha_I\right]\hat{a}^\dagger - \underbrace{\left[\alpha_R - i\alpha_I\right]}_{\alpha^*}\hat{a}\right)$$

$$= \exp\left(\alpha_R\left[\hat{a}^\dagger - \hat{a}\right] + i\alpha_I\left[\hat{a} + \hat{a}^\dagger\right]\right)$$

$$= \exp\left(-i\sqrt{2}\alpha_R\left[\frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i}\right] + i\sqrt{2}\alpha_I\left[\frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}\right]\right)$$

$$= \exp\left(i\left[\sqrt{2}\alpha_I\hat{x} - \sqrt{2}\alpha_R\hat{p}\right]\right)$$

point is to be careful w/ factor of $\sqrt{2}$ when going between conventions!

How to implement an n-mode displacement operator in the lab?

- Use an array of highly transmissive beamsplitters + strong local oscillators in coherent states. return to this point later.

What is the effect of displacement operator $\hat{D}_{\bar{r}}$ on mean vector of ρ?

The new mean vector is given by

$$\bar{r}' = \mathrm{Tr}\left[\hat{r}\,\hat{D}_{\bar{r}}\,\rho\,\hat{D}_{\bar{r}}^{\dagger}\right]$$

$$= \mathrm{Tr}\left[\hat{D}_{\bar{r}}^{\dagger}\,\hat{r}\,\hat{D}_{\bar{r}}\,\rho\right]$$

(When is this quantity finite?)

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So we've reduced the problem to computing $\hat{D}^{\dagger}_{\bar{r}} \, \hat{r} \, \hat{D}_{\bar{r}}$

which is the same as computing

$$\hat{D}^{\dagger}_{\bar{r}} \, \hat{x}_j \, \hat{D}_{\bar{r}} \quad \& \quad \hat{D}^{\dagger}_{\bar{r}} \, \hat{p}_j \, \hat{D}_{\bar{r}} \quad \forall j \in \{1, \ldots, n\}$$

Since $\hat{D}_{\bar{r}}$ is a tensor product of displacements, it follows that

$$\hat{D}^{\dagger}_{\bar{r}} \, \hat{x}_j \, \hat{D}_{\bar{r}} = \left(\hat{D}^{\dagger}_{\bar{r}_1} \otimes \cdots \otimes \hat{D}^{\dagger}_{\bar{r}_n}\right) \hat{x}_j \left(\hat{D}_{\bar{r}_1} \otimes \cdots \otimes \hat{D}_{\bar{r}_n}\right)$$

$$= \hat{D}^{\dagger}_{\bar{r}_j} \, \hat{x}_j \, \hat{D}_{\bar{r}_j}$$

& similar for $\hat{p}_j$.

To calculate this, we can employ BCH formula: (AKA Hadamard's lemma)

$$e^{\hat{X}} \hat{Y} e^{-\hat{X}} = \hat{Y} + [\hat{X}, \hat{Y}] + \frac{1}{2!} \cdot [\hat{X}, [\hat{X}, \hat{Y}]] + \frac{1}{3!} [\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]]$$

& we find that

$$\hat{D}^{+}_{\bar{r}_j}\, \hat{x}_j\, \hat{D}_{\bar{r}_j}$$

$$= e^{i[\bar{p}_j \hat{x}_j - \bar{x}_j \hat{p}_j]}\, \hat{x}_j\, e^{-i[\bar{p}_j \hat{x}_j - \bar{x}_j \hat{p}_j]}$$

$$= \hat{x}_j + [i(\bar{p}_j \hat{x}_j - \bar{x}_j \hat{p}_j), \hat{x}_j] + \ldots$$

$$= \hat{x}_j + i[-\bar{x}_j \hat{p}_j, \hat{x}_j]$$

$$= \hat{x}_j - i\bar{x}_j [\hat{p}_j, \hat{x}_j]$$

$$= \hat{x}_j - i\bar{x}_j(-i) = \hat{x}_j - \bar{x}_j$$

higher-order nested commutators vanish.

Similar calculation gives that

$$\hat{D}^{+}_{\bar{r}_j}\, \hat{p}_j\, \hat{D}_{\bar{r}_j} = \hat{p}_j - \bar{p}_j$$

$$\Rightarrow \hat{D}^{+}_{\bar{r}}\, \hat{r}\, \hat{D}_{\bar{r}} = \hat{r} - \bar{r}$$

$$\Rightarrow \bar{r}' = \mathrm{Tr}[\hat{D}^{+}_{\bar{r}}\, \hat{r}\, \hat{D}_{\bar{r}}\, \rho]$$

$$= \mathrm{Tr}[(\hat{r} - \bar{r})\rho]$$

$$= \underbrace{\mathrm{Tr}[\hat{r}\rho]}_{\text{original mean vector of } \rho} - \bar{r}$$

(So whole expression is finite when original mean vector of ρ is finite.)

How is the covariance matrix of ρ affected by a displacement?

Not affected ... To see this, consult definition of covariance matrix.

Displacement operator is the most general evolution realized by a Hamiltonian that is a real linear combination of the quadrature opeators.

What about a quadratic Hamiltonian?

A ~~Most~~ general form of such a Hamiltonian is as follows:

$$\hat{H} = \frac{1}{2} \hat{r}^T H \hat{r}$$

where H is a real, symmetric $2n \times 2n$ matrix.

- $\hat{H}$ is Hamiltonian operator & H is Hamiltonian matrix.

evolution realized by this Hamiltonian is

$$e^{-i\hat{H}t} = e^{-\frac{i}{2}\hat{r}^T H_t \hat{r}}$$

What is the effect on the mean vector of a state ρ?

Consider that

$$\bar{r}' = \mathrm{Tr}[\hat{r}\, e^{-i\hat{H}t} \rho\, e^{i\hat{H}t}]$$

$$= \mathrm{Tr}\{\underbrace{e^{i\hat{H}t}\, \hat{r}\, e^{-i\hat{H}t}}_{\text{so again figure out evolution in Heisenberg picture.}} \rho\}$$

Here we can again use the BCH formula

$$e^{\hat{X}} \hat{Y} e^{-\hat{X}} = \hat{Y} + [\hat{X},\hat{Y}] + \frac{1}{2!}[\hat{X},[\hat{X},\hat{Y}]] + \frac{1}{3!}[\hat{X},[\hat{X},[\hat{X},\hat{Y}]]] + \cdots$$

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then

$$e^{i\hat{H}t}\hat{r}e^{-i\hat{H}t}$$

$$= \hat{r} + [i\hat{H}t, \hat{r}] + \frac{1}{2!}[i\hat{H}t[i\hat{H}t, \hat{r}]] + \frac{1}{3!}[i\hat{H}t,[i\hat{H}t,[i\hat{H}t, \hat{r}]]] + \ldots$$

Consider that

$$[i\hat{H}t, \hat{r}]$$

$$= it[\hat{H}, \hat{r}]$$

Consider one @ a time

$$[\hat{H}, \hat{r}_\ell]$$

$$= \left[\frac{1}{2}\sum_{j,k} \hat{r}_j H_{jk} \hat{r}_k, \hat{r}_\ell\right]$$

$$= \frac{1}{2}\left(\sum_{j,k} H_{jk}(\hat{r}_j \hat{r}_k \hat{r}_\ell - \hat{r}_\ell \hat{r}_j \hat{r}_k)\right)$$

$$= \frac{1}{2}\left(\sum_{j,k} H_{jk}(\hat{r}_j \hat{r}_k \hat{r}_\ell - \hat{r}_j \hat{r}_\ell \hat{r}_k + \hat{r}_j \hat{r}_\ell \hat{r}_k - \hat{r}_\ell \hat{r}_j \hat{r}_k)\right)$$

$$= \frac{1}{2}\sum_{j,k} H_{jk}\left(\hat{r}_j [\hat{r}_k, \hat{r}_\ell] + [\hat{r}_j, \hat{r}_\ell]\hat{r}_k\right)$$

$$= \frac{1}{2} \sum_{jk} H_{jk} \left(\hat{r}_j i\Omega_{kl} + i\Omega_{jl} \hat{r}_k \right)$$

$$= \frac{i}{2} \sum_{jk} H_{jk} \hat{r}_j \Omega_{kl} + H_{jk} \Omega_{jl} \hat{r}_k$$

$$= \frac{i}{2} \sum_{jk} (-\Omega_{lk}) H_{kj} \hat{r}_j + (-\Omega_{lj}) H_{jk} \hat{r}_k$$

$$= -i \underbrace{\left[\Omega H \hat{r} \right]_l}_{\text{vector of operators}}$$

$$\Rightarrow \quad \left[i\hat{H}t, \hat{r} \right]$$

$$= it \left[\hat{H}, \hat{r} \right]$$

$$= it \left(-i \Omega H \hat{r} \right)$$

$$= \Omega H t \hat{r}$$

Now using linearity of commutator, we find that

$$\left[i\hat{H}t, \left[i\hat{H}t, \hat{r} \right] \right]$$

$$= \left[i\hat{H}t, \Omega H t \hat{r} \right] = (\Omega H t)^2 \hat{r}$$

& inductively

$$\underbrace{[i\hat{H}t, \ldots, [i\hat{H}t, \hat{r}] \cdots]}_{k \text{ times}} = (\Omega H t)^k \hat{r}$$

$$\Rightarrow e^{i\hat{H}t} \hat{r} e^{-i\hat{H}t} = \sum_{k=0}^{\infty} \frac{(\Omega H t)^k}{k!} \hat{r}$$

$$= e^{\Omega H t} \hat{r}$$

So the effect on the mean vector is

$$\bar{r}' = \mathrm{Tr}\left[e^{i\hat{H}t} \hat{r} e^{-i\hat{H}t} \rho\right]$$

$$= \mathrm{Tr}\left[e^{\Omega H t} \hat{r} \rho\right]$$

$$= e^{\Omega H t} \underbrace{\mathrm{Tr}[\hat{r} \rho]}_{\text{original mean vector}}$$

Also we have shown that
in Heisenberg picture, the
following holds

$$e^{+\frac{i}{2}\hat{r}^T H \hat{r} t} \; \hat{r} \; e^{-\frac{i}{2}\hat{r}^T H \hat{r} t}$$

$$= e^{\Omega H t} \hat{r}$$

What about commutation relations
for evolved operators?
natural for commutation relations to be
preserved.

That is, we should have for real, symmetric H

$$[e^{\Omega H}\hat{r}, (e^{\Omega H}\hat{r})^T] = e^{\Omega H} [\hat{r}, \hat{r}^T] (e^{\Omega H})^T$$

$$= e^{\Omega H} \, i\Omega (e^{\Omega H})^T$$

$$= i\Omega$$

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To see that the last equality is indeed true, consider that

$$e^{\Lambda H} i\Lambda \left(e^{\Lambda H}\right)^T$$

$$= e^{\Lambda H} \; i\Lambda \; e^{(\Lambda H)^T}$$

$$= e^{\Lambda H} \; i\Lambda \; e^{-H\Lambda}$$

$$= i\Lambda \; i\Lambda e^{\Lambda H} \; i\Lambda \; e^{-H\Lambda}$$

$$= i\Lambda \; e^{(i\Lambda)\Lambda H(i\Lambda)} \; e^{-H\Lambda}$$

$$= i\Lambda \; e^{(i\Lambda)(i\Lambda)H\Lambda} \; e^{-H\Lambda}$$

$$= i\Lambda \; e^{H\Lambda} \; e^{-H\Lambda}$$

$$= i\Lambda$$

so commutation relations preserved

any real matrix S for which $S\Lambda S^T = \Lambda$ is called symplectic

All symplectic matrices are invertible:

Consider that

$$S \Omega S^T = \Omega$$

$$\Rightarrow S \Omega S^T \Omega^T = \Omega \Omega^T = I$$

$$\Rightarrow S^{-1} = \Omega S^T \Omega^T = -\Omega S^T \Omega$$