Lecture 7 - Addendum

When ~~considering~~ defining a quadratic Hamiltonian, one could consider having it defined in terms of an arbitrary real $2n \times 2n$ matrix H. However, if there is any antisymmetric component of H, then the resulting "Hamiltonian" operator is not Hermitian.

- For this reason, it suffices to restrict our attention to real, symmetric $2n \times 2n$ Hamiltonian matrices when defining the Hamiltonian operator.

To see this, let H be an arbitrary $2n \times 2n$ real matrix. It can be written as

$$H = \frac{H + H^T}{2} + \frac{H - H^T}{2}$$

$$\equiv \underbrace{H^s}_{\text{symmetric part}} + \underbrace{H^a}_{\text{antisymmetric part}}$$

Now consider the operator

$$\frac{1}{2}\hat{r}^T H \hat{r} = \frac{1}{2}\hat{r}^T \left(H^s + H^a\right)\hat{r}$$

$$= \frac{1}{2}\hat{r}^T H^s \hat{r} + \frac{1}{2}\hat{r}^T H^a \hat{r}$$

Focus on the 2^{nd} term

$$\frac{1}{2}\hat{r}^T H^a \hat{r} = \frac{1}{2}\sum_{jk} \hat{r}_j H^a_{jk} \hat{r}_k$$

③

$$= \frac{1}{2} \sum_{j<k} \hat{r}_j^* H^a_{jk} \hat{r}_k + \hat{r}_k H^a_{kj} \hat{r}_j$$

$$= \frac{1}{2} \sum_{j<k} \hat{r}_j H^a_{jk} \hat{r}_k - \hat{r}_k H^a_{jk} \hat{r}_j$$ (diagonal part of H^a is equal to zero)

$$= \frac{1}{2} \sum_{j<k} H^a_{jk} (\hat{r}_j \hat{r}_k - \hat{r}_k \hat{r}_j)$$

$$= \frac{1}{2} \sum_{j<k} H^a_{jk} [\hat{r}_j, \hat{r}_k]$$

$$= \frac{1}{2} \sum_{j<k} H^a_{jk} \, i \Lambda_{jk}$$

$$= i \cdot \frac{1}{2} \sum_{j<k} H^a_{jk} \Lambda_{jk}$$

$$\equiv i c \qquad \text{where } c \in \mathbb{R}$$

Thus, we find that if $H^a \neq 0$, then

$$\frac{1}{2} \hat{r}^T H \hat{r} = \frac{1}{2} \hat{r}^T H^s \hat{r} + i c$$

& it is thus not possible for $\frac{1}{2} \hat{r}^T H \hat{r}$ to be Hermitian (& a legitimate Hamiltonian operator)

In class, we proved that

if $S = e^{\Omega H}$ for H real, symmetric $2n \times 2n$,

then S is symplectic, i.e.,

$$S \Omega S^T = \Omega .$$

Here, we show that if

$H = \Omega^T \ln S$, for ln the matrix logarithm,

then H is symmetric.

Consider that

$$\begin{aligned} H^T &= (\Omega^T \ln S)^T \\ &= (\ln S^T)\, \Omega \\ &= \Omega\, \Omega^T (\ln S^T) \Omega \\ &= \Omega \ln(\Omega^T S^T \Omega) \end{aligned}$$

From $S\Omega S^T = \Omega$ it follows that

$$\underbrace{S\Omega S^T}_{S^{-1}}\Omega^T = \Omega\Omega^T = I$$

then

$$\begin{aligned}
&\Omega \ln(\Omega^T S^T \Omega) \\
&= \Omega \ln\left((-\Omega^T) S^T (-\Omega)\right) \\
&= \Omega \ln(\Omega S^T \Omega^T) \\
&= \Omega \ln S^{-1} \\
&= -\Omega \ln S \\
&= \Omega^T \ln S
\end{aligned}$$

$\Rightarrow$ $H = H^T$ & H is real & symmetric.

Thus, from any symplectic matrix S, we can get its Hamiltonian matrix from $H = \Omega^T \ln S$