Lecture 6

Recall canonical operators

$(\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_m, \hat{p}_m)$

For a state ρ of multiple modes,

the mean vector is given by

$$\bar{x}_1 = \mathrm{Tr}[\hat{x}_1 \rho] = \mathrm{Tr}[(\hat{x}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I})\rho]$$

$$\bar{p}_1 = \mathrm{Tr}[\hat{p}_1 \rho] = \mathrm{Tr}[(\hat{p}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I})\rho]$$

$$\bar{x}_j = \mathrm{Tr}[\hat{x}_j \rho] = \langle \hat{x}_j \rangle_\rho$$

$$\bar{p}_j = \mathrm{Tr}[\hat{p}_j \rho] = \langle \hat{p}_j \rangle_\rho$$

can write this in a shorthand as

$\bar{r} = \mathrm{Tr}[\hat{r}\rho]$ where we recall

$$\hat{r} = \begin{pmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \vdots \\ \hat{x}_m \\ \hat{p}_m \end{pmatrix} \quad \text{\& define } \mathrm{Tr}[\hat{r}\rho] = \begin{pmatrix} \mathrm{Tr}[\hat{x}_1\rho] \\ \mathrm{Tr}[\hat{p}_1\rho] \\ \vdots \\ \vdots \end{pmatrix}$$

A state need not have a finite mean

(just like classical prob. dist's need not have a finite mean.)

An important case of states w/ mean vector existing:

Consider total photon number operator $\hat{N} = \sum_{j=1}^{m} \hat{n}_j$

where $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$

Suppose that $\mathrm{Tr}[\hat{N}\rho] < \infty$

(finite-energy state)

$\Rightarrow \mathrm{Tr}[\hat{n}_j \rho] < \infty$

Then by applying the fact that

$\mathrm{Tr}[\hat{n}_j \rho] = \mathrm{Tr}[(\hat{x}_j^2 + \hat{p}_j^2 - 1)\rho] < \infty$

$\Rightarrow \mathrm{Tr}[\hat{x}_j^2 \rho], \mathrm{Tr}[\hat{p}_j^2 \rho] < \infty$ (3)

& Cauchy-Schwarz

$$\left|\mathrm{Tr}[A^\dagger B]\right| \leq \|A\|_2 \, \|B\|_2$$

we conclude that

$$|\bar{x}_j| = \left|\mathrm{Tr}[\hat{x}_j \rho]\right| = \left|\mathrm{Tr}[\hat{x}_j \sqrt{\rho}\sqrt{\rho}]\right|$$

$$\leq \sqrt{\mathrm{Tr}[\hat{x}_j \sqrt{\rho}\sqrt{\rho}\hat{x}_j] \cdot \mathrm{Tr}[\sqrt{\rho}\sqrt{\rho}]}$$

$$= \sqrt{\mathrm{Tr}[\hat{x}_j^2 \rho]} < \infty$$

Similarly, $|\bar{p}_j| = \left|\mathrm{Tr}[\hat{p}_j \rho]\right| < \infty$.

Finite-energy states are the physically realistic ones, and they have finite means.

Also critical for analysis of bosonic states is the covariance matrix. (when it exists)

Calling the elements of $\hat{r}$ as $\hat{r}_j$ where $j \in \{1, \ldots, 2m\}$, entries of covariance matrix are given by

$$\sigma_{jk} = \mathrm{Tr}\left[\left(\hat{r}_j^c \hat{r}_k^c + \hat{r}_k^c \hat{r}_j^c\right) \rho\right]$$
$$= \mathrm{Tr}\left[\{\hat{r}_j^c, \hat{r}_k^c\} \rho\right]$$

where $\hat{r}_j^c$ is the <u>centered</u> version of $\hat{r}_j$:

$$\hat{r}_j^c = \hat{r}_j - \langle \hat{r}_j \rangle_\rho$$
$$= \hat{r}_j - \bar{r}_j$$

Also $\sigma_{jk} = \langle \{\hat{r}_j^c, \hat{r}_k^c\} \rangle_\rho$

& observe $\sigma_{jk} \in \mathbb{R}$.

The covariance matrix entries also need not be finite in general, but they are finite for a finite-energy state.

To see this, 1st consider the case for $j=k$ in σ_{jk}:

$$\sigma_{jj} = \langle \{ \hat{r}_j^c , \hat{r}_j^c \} \rangle_\rho$$

$$= 2 \langle (\hat{r}_j^c)^2 \rangle_\rho$$

$$= 2 \left(\langle \hat{r}_j^2 \rangle_\rho - \langle \hat{r}_j \rangle_\rho^2 \right)$$

this is finite by previous argument.

already argued to be finite for finite-energy state

What about when $j \neq k$ in σ_{jk}?

$$|\sigma_{jk}| = \left| \langle \hat{r}_j^c \hat{r}_k^c + \hat{r}_k^c \hat{r}_j^c \rangle_\rho \right|$$

$$= \left| \langle \hat{r}_j^c \hat{r}_k^c \rangle_\rho + \langle \hat{r}_k^c \hat{r}_j^c \rangle_\rho \right|$$

$$\leq \left| \langle \hat{r}_j^c \hat{r}_k^c \rangle_\rho \right| + \left| \langle \hat{r}_k^c \hat{r}_j^c \rangle_\rho \right|$$

$$\left| \langle \hat{r}_j^c \hat{r}_k^c \rangle_\rho \right| = \left| \mathrm{Tr}[\hat{r}_j^c \hat{r}_k^c \rho] \right|$$

$$= \left| \mathrm{Tr}[\sqrt{\rho}\, \hat{r}_j^c \hat{r}_k^c \sqrt{\rho}] \right|$$

$$\leq \sqrt{\mathrm{Tr}[(\hat{r}_j^c)^2 \rho]\, \mathrm{Tr}[(\hat{r}_k^c)^2 \rho]}$$

$$= \sqrt{\left(\mathrm{Tr}[\hat{r}_j^2 \rho] - \mathrm{Tr}[\hat{r}_j \rho]^2\right) \times \left(\mathrm{Tr}[\hat{r}_k^2 \rho] - \mathrm{Tr}[\hat{r}_k \rho]^2\right)}$$

$$< \infty.$$

$\Rightarrow |\sigma_{jk}| < \infty$ for a finite-energy state.

We just proved that a finite-energy state has a finite covariance matrix. Is the reverse implication true?

yes. Suppose entries of covariance matrix are finite. (In fact, if diagonal elements of σ are finite, then all are.)

Then $$\mathrm{Tr}[\hat{N}\rho] = \sum_{j=1}^{m} \mathrm{Tr}[\hat{n}_j \rho]$$

$$= \sum_{j=1}^{m} \left(\mathrm{Tr}[\hat{x}_j^2 \rho] + \mathrm{Tr}[\hat{p}_j^2 \rho] - 1 \right) < \infty.$$

$\boxed{\text{So a state has finite energy iff mean vector \& covariance matrix are finite.}}$

Rather than writing out all $2m \times 2m$ entries of the covariance matrix, we can use an abbreviated notation for it as

$$\sigma = \mathrm{Tr}\left[\left\{(\hat{r}-\bar{r}),(\hat{r}-\bar{r})^{\dagger}\right\}\rho\right]$$

where this is shorthand for

$$\left\{(\hat{r}-\bar{r}),(\hat{r}-\bar{r})^{\dagger}\right\} = \begin{bmatrix} \{\hat{r}_1-\bar{r}_1, \hat{r}_1-\bar{r}_1\} & \{\hat{r}_1-\bar{r}_1, \hat{r}_2-\bar{r}_2\} & \cdots \\ \{\hat{r}_2-\bar{r}_2, \hat{r}_1-\bar{r}_1\} & \ddots & \vdots \\ \cdots & \cdots & \ddots \end{bmatrix}$$

& trace w/ ρ is taken as

$$\begin{bmatrix} \mathrm{Tr}[\{\hat{r}_1-\bar{r}_1, \hat{r}_1-\bar{r}_1\}\rho] & \cdots \\ \mathrm{Tr}[\{\hat{r}_2-\bar{r}_2, \hat{r}_1-\bar{r}_1\}\rho] & \ddots \\ & & \ddots \end{bmatrix}$$

~~What other constraints should a quantum covariance matrix satisfy?~~

Classically, a covariance matrix for a vector of RVs is ~~symmetric~~ Hermitian & positive semidefinite.

~~Symmetric~~ Hermitian property follows by definition. Proof of PSD is as follows:

Let $\underline{X}$ be a random vector w/ values in $\mathbb{C}^m$. then ~~CM is defined as~~

$$\Sigma = \mathbb{E}\left[(\underline{X} - \mathbb{E}[\underline{X}])(\underline{X} - \mathbb{E}[\underline{X}])^\dagger\right]$$

Let $\underline{w}$ be a constant vector in $\mathbb{C}^m$.

Then $\underline{w}^T \Sigma \underline{w} = \underline{w}^T \mathbb{E}\left[(\underline{X} - \mathbb{E}[\underline{X}])(\underline{X} - \mathbb{E}[\underline{X}])^\dagger\right] \underline{w}$

$$= \mathbb{E}\left[\underline{w}^\dagger(\underline{X}-\mathbb{E}[\underline{X}])(\underline{X}-\mathbb{E}[\underline{X}])^\dagger\underline{w}\right]$$

$$= \mathbb{E}\left[\left|\underline{w}^\dagger(\underline{X}-\mathbb{E}[\underline{X}])\right|^2\right] \geq 0$$

Since this holds $\forall\, \underline{w} \in \mathbb{C}^m$,

$$\Rightarrow \Sigma \geq 0 \qquad (\Sigma \text{ is P.S.D.})$$

We can actually use a very similar argument to establish an uncertainty principle constraint on a quantum covariance matrix:

~~Consider~~

$$\boxed{\sigma + i\Omega \geq 0}$$

Note that $\sigma + i\Omega$ has complex entries

Consider the $2m \times 2m$ complex matrix given by

$$\tau = 2\,\mathrm{Tr}\Big[\overbrace{(\hat{r}-\bar{r})(\hat{r}-\bar{r})^\dagger}^{2m\times 2m \text{ matrix of operators}}\rho\Big]$$

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Note that this is different from the q. covariance matrix

We first prove that

τ is PSD + then deduce

the statement of the theorem

Let $\underline{w} \in \mathbb{C}^{2m}$

then $\underline{w}^\dagger \tau \underline{w}$

$$= 2 \underline{w}^\dagger \operatorname{Tr}[(\hat{r} - \bar{r})(\hat{r} - \bar{r})^\dagger \rho] \underline{w}$$

$$= 2 \operatorname{Tr}[\underline{w}^\dagger (\hat{\underline{r}} - \bar{r})(\hat{r} - \bar{r})^\dagger \underline{w} \rho]$$

Consider that

$$\hat{B} \equiv \underline{w}^\dagger (\hat{r} - \bar{r}) = \sum_{j=1}^{2m} w_j^* (\hat{r}_j - \bar{r}_j)$$

then the above equals

$$2 \operatorname{Tr}[\hat{B} \hat{B}^\dagger \rho]$$

The operator $\hat{B}\hat{B}^\dagger$ is PSD &

so is ρ $\Rightarrow$ $\operatorname{Tr}[\hat{B}\hat{B}^\dagger \rho] \geq 0$

Since this holds $\forall\, \underline{w} \in \mathbb{C}^{2m}$, conclude that τ is P.S.D.

Now, consider that

$$2 \hat{r}_j \hat{r}_k = \{\hat{r}_j, \hat{r}_k\} + [\hat{r}_j, \hat{r}_k]$$

$$\Rightarrow 2(\hat{r} - \bar{r})(\hat{r} - \bar{r})^{+}$$

$$= \{(\hat{r} - \bar{r}), (\hat{r} - \bar{r})^{+}\} + [(\hat{r} - \bar{r}), (\hat{r} - \bar{r})^{+}]$$

$$= \{(\hat{r} - \bar{r}), (\hat{r} - \bar{r})^{+}\} + [\hat{r}, \hat{r}^{+}]$$

$$\Rightarrow \tau = 2\, \mathrm{Tr}[(\hat{r} - \bar{r})(\hat{r} - \bar{r})^{+} \rho]$$

$$= \mathrm{Tr}[\{(\hat{r} - \bar{r}), (\hat{r} - \bar{r})^{+}\} \rho]$$

$$+ \mathrm{Tr}[[\hat{r}, \hat{r}^{+}] \rho]$$

$$= \sigma + i \Omega$$

$$\tau \geq 0 \Rightarrow \sigma + i \Omega \geq 0$$

$\square$

Note that the eigenvalues of a matrix do not change under a transpose. So if they are positive, then they remain positive.

Then $\sigma + i\Omega \geq 0$ (1)

$\Rightarrow \quad (\sigma + i\Omega)^T \geq 0$

$\Rightarrow \quad \sigma + i\Omega^T \geq 0$

$\Rightarrow \quad \sigma - i\Omega \geq 0$ (2)

Adding (1) + (2) then gives $2\sigma \geq 0$

$\Rightarrow \quad \sigma \geq 0$

So every q. covariance matrix is P.S.D.

However, it is in fact the case that any q. covariance matrix is <u>positive definite</u>

This makes them more special & easier to work w/ mathematically than classical covariance matrices.

<u>Proof</u>: W/ the goal of arriving @ a contradiction, suppose that q. cov. matrix σ is not positive definite. That is, $\exists$ a real, non-zero vector $Y \in \mathbb{R}^{2m}$ such that

$\sigma \Psi = 0$. Then for $\varepsilon \in \mathbb{R}$,

set $\Psi(\varepsilon) = (I + \varepsilon i\Omega)\Psi$.

Using that, by assumption, $\sigma\Psi = 0$

$\Psi^T \Omega \Psi = 0 \quad \forall \Psi \in \mathbb{R}^{2m}$, &

$(i\Omega)^2 = I$, we find that

$$\Psi(\varepsilon)^\dagger (\sigma + i\Omega)\Psi(\varepsilon)$$

$$= \left[(I + \varepsilon i\Omega)\Psi\right]^\dagger (\sigma + i\Omega)\left[(I + \varepsilon i\Omega)\Psi\right]$$

$$= \Psi^T (I + \varepsilon i\Omega)(\sigma + i\Omega)(I + \varepsilon i\Omega)\Psi$$

$$= \Psi^T (I + \varepsilon i\Omega)(\sigma + i\Omega + \varepsilon\sigma i\Omega + \varepsilon I)\Psi$$

$$= \Psi^T (I + \varepsilon i\Omega)(i\Omega + \varepsilon\sigma i\Omega + \varepsilon I)\Psi$$

$$= \Psi^T \big(i\Omega + \varepsilon\sigma i\Omega + \varepsilon I$$
$$+ \varepsilon i\Omega(i\Omega + \varepsilon\sigma i\Omega + \varepsilon I)\big)\Psi$$

$$= \psi^T \left(i\Lambda + \varepsilon \sigma i \Lambda + 2\varepsilon I + \varepsilon^2 \Lambda^T \sigma \Lambda + \varepsilon^2 i \Lambda \right) \psi$$

$$= \psi^T \left(2\varepsilon I + \varepsilon^2 \Lambda^T \sigma \Lambda \right) \psi$$

$$= 2\varepsilon \psi^T \psi + \varepsilon^2 [\Lambda \psi]^T \sigma [\Lambda \psi]$$

suppose $[\Lambda\psi]^T \sigma \Lambda \psi = 0$.

then picking $\varepsilon < 0$

$\Rightarrow \quad 2\varepsilon \psi^T \psi < 0$

which contradicts the fact that

$$\psi(\varepsilon)^\dagger (\sigma + i\Lambda)\psi(\varepsilon) \geq 0$$

(i.e., $\sigma + i\Lambda$ is PSD)

Suppose now that $[\Lambda\psi]^T \sigma \Lambda \psi > 0$.

then pick

$\varepsilon < 0$ + such that

$$|\varepsilon| < \frac{2\psi^T\psi}{[\Lambda\psi]^T \sigma \Lambda\psi}$$

$\Rightarrow$

$$2\varepsilon\psi^T\psi + \varepsilon^2 [\Lambda\psi]^T \sigma \Lambda\psi < 0$$

$\Rightarrow$ $\exists$ $\psi(\varepsilon)$ such that

$$\psi(\varepsilon)^\dagger (\sigma + i\Lambda) \psi(\varepsilon) < 0$$

again contradicting assumption that

$$\sigma + i\Lambda \geq 0$$

$\lightning$ $\Rightarrow$ must be the case that σ is <u>positive</u> <u>definite</u>.

Uncertainty principle for a single-mode bosonic state

$$\sigma = \begin{bmatrix} 2\langle (\hat{x}^c)^2 \rangle_\rho & \langle \{\hat{x}^c, \hat{p}^c\} \rangle_\rho \\ \langle \{\hat{x}^c, \hat{p}^c\} \rangle_\rho & 2\langle (\hat{p}^c)^2 \rangle_\rho \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$\sigma + i\Omega \geq 0 \iff \mathrm{Det}(\sigma) \geq 1$ &
$\sigma > 0$

one direction is simple, given what we have shown

$\sigma + i\Omega \geq 0 \Rightarrow \sigma > 0$

$\sigma + i\Omega \geq 0 \Rightarrow \mathrm{Det}(\sigma + i\Omega) \geq 0$

$$\sigma + i\Omega = \begin{bmatrix} \sigma_{11} & \sigma_{21} + i \\ \sigma_{21} - i & \sigma_{22} \end{bmatrix}$$

$$\Rightarrow \mathrm{Det}(\cdot\cdot) = \sigma_{11}\sigma_{22} - (\sigma_{21} + i)(\sigma_{21} - i)$$
$$= \sigma_{11}\sigma_{22} - (\sigma_{21}^2 + 1) \geq 0 \Rightarrow \mathrm{Det}(\sigma) \geq 1$$