Lecture 5

- We now work w/ a single mode bosonic system.
- What is meant by a single mode? Mathematically, a separable Hilbert space equipped w/ canonical operators.
- Physically, a bosonic quantum system. For light, a mode is well defined in space, frequency, polarization, & time.

Let us begin w/ photon number states

$$\{|n\rangle\}_{n=0}^{\infty}.$$

Mathematically, these are the same as the Kronecker functions discussed previously.

Physically, they correspond to a bosonic system in a state of definite photon number.

- Any state can be represented in terms of photon-number basis. Then define the annihilation & creation operators ~~as~~ in terms of the following action on photon-number basis:

annihilation: $$\boxed{\begin{aligned} \hat{a}|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}|0\rangle &= 0 \end{aligned}}$$ for $n \geq 1$

From this, we deduce that its matrix elements are

$$\langle m|\big(\hat{a}|n\rangle\big) = \sqrt{n}\,\langle m|n-1\rangle$$
$$= \sqrt{n}\,\delta_{m,n-1} \quad \text{for } n \geq 1$$

& $$\langle m|\big(\hat{a}|0\rangle\big) = 0 \quad \text{for } n=0.$$

The annihilation operator is <u>unbounded</u> b/c

$$\sup_{\|\psi\| = \|\phi\| = 1} |\langle \psi|\hat{a}\phi\rangle| = \infty$$

Choose $|\psi\rangle = |n-1\rangle$ &

$|\phi\rangle = |n\rangle$ to get

$\left|\langle\psi|\tilde{a}\,\phi\rangle\right| = \sqrt{n}$ & then

take $\lim\limits_{n\to\infty}$.

Adjoint of $\hat{a}$ is defined from

$$\langle m|\left(\hat{a}^{+}|n\rangle\right) = \left(\tilde{a}|m\rangle\right)^{+}|n\rangle$$

$$= \sqrt{m}\,\langle m-1|n\rangle$$

$$= \sqrt{m}\,\delta_{m-1,n}$$

By setting $m = n+1$, we find that

$$\langle n+1|\left(\hat{a}^{+}|n\rangle\right) = \sqrt{n+1}\,\delta_{n,n}$$
$$= \sqrt{n+1}$$

& since $\langle m|\left(\hat{a}^{+}|n\rangle\right)$ for $m \neq n+1$,

we conclude that

$$\boxed{\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle}$$

$\Rightarrow$ creation operator.

This is also unbounded, by a similar argument.

From these relations, we can then deduce canonical commutation relations: (CCR)

$$[\hat{a}, \hat{a}^{+}] = I$$

Consider the action on photon-number basis

$$[\hat{a}, \hat{a}^{+}] |n\rangle$$

$$= (\hat{a}\hat{a}^{+} - \hat{a}^{+}\hat{a}) |n\rangle$$

$$= \hat{a}\hat{a}^{+} |n\rangle - \hat{a}^{+}\hat{a} |n\rangle$$

$$= \sqrt{n+1}\ \hat{a} |n+1\rangle - \sqrt{n}\, \hat{a}^{+} |n-1\rangle$$

$$= \sqrt{n+1}\sqrt{n+1} |n\rangle - \sqrt{n}\sqrt{n} |n\rangle$$

$$= (n+1-n) |n\rangle$$

$$= |n\rangle \qquad \text{holds for all } n \geq 0$$

$$\Rightarrow \quad [\hat{a}, \hat{a}^{+}] |n\rangle = |n\rangle \quad \forall n \geq 0$$

Since this holds for an O.N. basis, we conclude that $[\hat{a}, \hat{a}^{+}] = I$

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Using the facts that

$$[\hat{a}^\dagger, \hat{a}] = -[\hat{a}, a^\dagger] = -I$$

$$\& \; [\hat{a}, \hat{a}] = 0,$$

$$[\hat{a}^\dagger, \hat{a}^\dagger] = 0,$$

we can capture commutation relations in a matrix as

$$\left[\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}, (\hat{a}^\dagger \hat{a}) \right]$$

$$= \begin{bmatrix} [\hat{a}, \hat{a}^\dagger] & [\hat{a}, \hat{a}] \\ [\hat{a}^\dagger, \hat{a}^\dagger] & [\hat{a}^\dagger, \hat{a}] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \hat{I}.$$

$$= \sigma_z \otimes \hat{I}$$

Also, consider that

$$\hat{a}^\dagger \hat{a} |n\rangle = \sqrt{n} \; \hat{a}^\dagger |n-1\rangle$$

$$= n |n\rangle$$

so that photon-number states $|n\rangle$ are eigenstates of $\hat{a}^\dagger \hat{a} = \hat{n}$ w/ eigenvalue n. Write $\hat{n} = \sum_{n=0}^{\infty} n|n\rangle\langle n|$

Now define position & momentum quadrature operators as

$$\hat{x} = \frac{\hat{a} + \hat{a}^{+}}{\sqrt{2}} \quad \text{or} \quad \begin{bmatrix} \hat{x} \\ \hat{p} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1/i & -1/i \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{a}^{+} \end{bmatrix}$$

Hermitian

$$\hat{p} = \frac{\hat{a} - \hat{a}^{+}}{\sqrt{2}\, i} \qquad = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{a}^{+} \end{bmatrix}$$

$$\Rightarrow \quad \hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}}$$

$$\hat{a}^{+} = \frac{\hat{x} - i\hat{p}}{\sqrt{2}}$$

or

$$\begin{bmatrix} \hat{a} \\ \hat{a}^{+} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{p} \end{bmatrix}$$

$\hat{x}$ & $\hat{p}$ are <u>unbounded</u> since $\hat{a}$ & $\hat{a}^{+}$ are.

From commutation relations of $\begin{pmatrix}\hat{a}\\ \hat{a}^+\end{pmatrix}$, we can work out those for $\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}$ as

$$\begin{bmatrix} [\hat{x},\hat{x}] & [\hat{x},\hat{p}] \\ [\hat{p},\hat{x}] & [\hat{p},\hat{p}] \end{bmatrix} = \left[\begin{pmatrix}\hat{x}\\ \hat{p}\end{pmatrix}, \begin{pmatrix}\hat{x} & \hat{p}\end{pmatrix}\right]$$

$$= \left[\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}\hat{a}\\ \hat{a}^+\end{pmatrix}, \begin{pmatrix}\hat{a}^+ & \hat{a}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\left[\begin{pmatrix}\hat{a}\\ \hat{a}^+\end{pmatrix}, \begin{pmatrix}\hat{a}^+ & \hat{a}\end{pmatrix}\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes\hat{I}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}$$

$$= i\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\otimes\hat{I}$$

Eigenvectors of $\hat{x}$, $\hat{p}$, $\hat{a}$, $\hat{a}^+$

It is impossible for $\hat{x}$ to have normalized eigenvectors.

Suppose that $|\psi\rangle$ is an eigenvector of $\hat{x}$ w/ eigenvalue $\lambda \in \mathbb{R}$, i.e.,

$$\hat{x}|\psi\rangle = \lambda|\psi\rangle$$

Then we can define

$$\hat{x}' = \hat{x} - \lambda I$$

An eigenvector of this w/ zero eigenvalue is $|\psi\rangle$ b/c

$$\hat{x}'|\psi\rangle = \hat{x}|\psi\rangle - \lambda|\psi\rangle = \underbrace{\lambda|\psi\rangle - \lambda|\psi\rangle}_{=0}$$

Consider also that

$$[\hat{x}', \hat{p}] = \hat{x}'\hat{p} - \hat{p}\hat{x}'$$

$$= (\hat{x} - \lambda I)\hat{p} - \hat{p}(\hat{x} - \lambda I)$$

$$= \hat{x}\hat{p} - \hat{p}\hat{x} = iI$$

Now consider that

$$\langle\psi| [\hat{x}', \hat{p}] |\psi\rangle = \langle\psi| \hat{x}'\hat{p} - \hat{p}\hat{x}' |\psi\rangle$$

$$= \langle\psi| \lambda\hat{p} - \lambda\hat{p} |\psi\rangle$$

$$= 0$$

However $\langle\psi| iI |\psi\rangle = i$

which contradicts equality

$$[\hat{x}', \hat{p}] = iI$$

$\Rightarrow$ $\hat{x}$ cannot have a normalized eigenvector.

By same argument, $\hat{p}$ cannot have a normalized eigenvector.

What about creation operator $\hat{a}^+$?

Suppose that there exists a normalizable $|\psi\rangle$ such that

(*) $\hat{a}^+|\psi\rangle = \mu|\psi\rangle$ for $\mu \in \mathbb{C}$.

We can write $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

for $c_n \in \mathbb{C}$ such that $\sum_{n=0}^{\infty} |c_n|^2 = 1$

Then it follows from (*) that

$$\sum_{n=0}^{\infty} c_n a^+|n\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n+1}\,|n+1\rangle$$

$$= \sum_{n=0}^{\infty} c_n \mu |n\rangle$$

$\Rightarrow c_0\mu = 0, \quad c_0 = c_1\mu, \quad c_1\sqrt{2} = c_2\mu,$ ...

If $\mu \neq 0$, then $c_0 = 0 \Rightarrow c_1 = 0$,
..... etc.

If $\mu = 0$, then $c_0 = 0$ from 2nd eq.,
$c_1 = 0$ from 3rd, etc.

so $\hat{a}^{\dagger}$ does not have eigenvectors?

What about $\hat{a}$?

eigenstates are coherent states,
each of which is parametrized
by $\alpha \in \mathbb{C}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \qquad \left(\text{eigen value-vector relation}\right)$$

Proof:

Expand $|\alpha\rangle$ in terms of
number basis as

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \qquad \text{for } c_n \in \mathbb{C}$$

Now apply $\hat{a}$ to $|\alpha\rangle$

$$\hat{a}|\alpha\rangle = \hat{a}\sum_{n=0}^{\infty} c_n |n\rangle$$

$$= \sum_{n=0}^{\infty} c_n \,\hat{a}\,|n\rangle$$

$$= \sum_{n=1}^{\infty} c_n \sqrt{n}\,|n-1\rangle$$

to be an eigenstate, we should have that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad \text{or}$$

$$\sum_{n=1}^{\infty} c_n \sqrt{n}\,|n-1\rangle = \sum_{n=0}^{\infty} \alpha\, c_n |n\rangle$$

Equating coefficients term by term gives the recursion relation

$$c_n\sqrt{n} = \alpha\, c_{n-1}$$

or $$c_n = \frac{\alpha}{\sqrt{n}}\, c_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}}\, c_{n-2}$$

$$= \cdots = \frac{\alpha^n}{\sqrt{n!}}\, c_0$$

so that

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

From normalization requirement, we find that

$$1 = \langle\alpha|\alpha\rangle = |c_0|^2 \sum_{n,n'=0}^{\infty} \frac{\alpha^{*n} \alpha^{n'}}{\sqrt{n!n'!}} \langle n|n'\rangle$$

$$= |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2}$$

$$\Rightarrow \quad c_0 = \exp(-\tfrac{1}{2}|\alpha|^2)$$

$$\Rightarrow |\alpha\rangle = e^{-1/2|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Multiple modes

By tensoring together several separable Hilbert spaces, each corresponding to a bosonic mode, we get multiple-mode bosonic Hilbert space

Each mode j is equipped w/ canonical operators $\hat{x}_j$ + $\hat{p}_j$ for $j \in \{1, \ldots, n\}$.

if $j \neq k$ then $[\hat{x}_j, \hat{p}_k] = 0$

b/c these operators are acting on different Hilbert spaces or particles that are distinguishable (by spatial, temporal, polarization, or frequency degree of freedom).

So then canonical commutation relations are encoded by

$$[\hat{x}_j, \hat{p}_k] = i\delta_{jk}\hat{I}$$

helpful to introduce a vector of quadrature operators

$$\hat{r} = \begin{pmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \vdots \\ \hat{x}_n \\ \hat{p}_n \end{pmatrix}$$

Then the CCR are encoded in the following matrix:

$$[\hat{r}, \hat{r}^{\dagger}] = \left[\begin{pmatrix} \hat{x}_1 \\ \hat{p}_1 \\ \vdots \\ \hat{x}_n \\ \hat{p}_n \end{pmatrix}, (\hat{x}_1, \hat{p}_1 \cdots \hat{x}_n \hat{p}_n) \right]$$

$$= \begin{bmatrix} [\hat{x}_1, \hat{x}_1] & [\hat{x}_1, \hat{p}_1] & \cdots \\ [\hat{p}_1, \hat{x}_1] & [\hat{p}_1, \hat{p}_1] & \\ \vdots & & \ddots \end{bmatrix}$$

$$= i\Omega$$

(could write as $i\Omega \otimes I$)

where $\Lambda = \bigoplus_{j=1}^{n} \Lambda_1$ w/ $\Lambda_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$= I_n \otimes \Lambda_1$$

Λ is a special matrix, called symplectic form, which realizes a symplectic inner product via

$$x^T \Lambda y$$

Λ is antisymmetric:

$$\begin{aligned} \Lambda^T &= (I_n \otimes \Lambda_1)^T \\ &= I_n \otimes \Lambda_1^T \\ &= I_n \otimes (-\Lambda_1) \\ &= -(I_n \otimes \Lambda_1) \\ &= -\Lambda . \end{aligned}$$

Observe that $\Lambda_1^2 = -I$

$\Rightarrow$ Λ is an orthogonal matrix

$$\Lambda^T \Lambda = I_{2n}$$

Commutator matrix $i\Omega$ is involutory:

$$(i\Omega)^2 = I$$

(each of these facts gets used a lot)

One can also use a different order for vector of operators as

$$\hat{s} = \begin{bmatrix} \hat{x}_1 \\ \vdots \\ \hat{x}_n \\ \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{bmatrix}$$

In this case,

$$[\hat{s}, \hat{s}^\dagger] = iJ \quad \text{where}$$

$$J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0 \end{bmatrix} = \Omega_1 \otimes I_n$$

can also write vector of

annihilation & creation operators as

$$\underline{\hat{a}} = \begin{bmatrix} \hat{a}_1 \\ \hat{a}_1^+ \\ \vdots \\ \hat{a}_n \\ \hat{a}_n^+ \end{bmatrix}$$

Then $\quad \underline{\hat{a}} = U \hat{r}$

where unitary U is

$$U = \bigoplus_{j=1}^{n} u$$

$$= I_n \otimes u$$

w/ $u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$

then $\quad [\underline{\hat{a}}, \underline{\hat{a}}^+] = [U\hat{r}, \hat{r}^+ U^+]$

$$= U [\hat{r}, \hat{r}^+] U^+$$

$$= \mathcal{U}\, i\,(I_n \otimes \Lambda_1)\,\mathcal{U}^{\dagger}$$

$$= i\,(I_n \otimes u)(I_n \otimes \Lambda_1)(I_n \otimes u^{\dagger})$$

$$= i\, I_n \otimes u \Lambda_1 u^{\dagger}$$

$$= I_n \otimes \sigma_z = \bigoplus_{i=1}^{n} \sigma_z$$

$$\left(u \Lambda_1 u^{\dagger} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \right)$$

$$\Rightarrow \boxed{[\underline{\hat{a}}, \underline{\hat{a}}^{\dagger}] = I_n \otimes \sigma_z}$$

One could even define another order

$$\begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \\ \hat{a}_1^{\dagger} \\ \vdots \\ \hat{a}_n^{\dagger} \end{bmatrix},$$ but we won't write it.

Indeed, all four conventions are found in the literature.