Lecture 4

Different notions of convergence in ∞-dim. case.

- Given that we have been working w/ normed spaces, one might think that a natural notion of convergence for a sequence $\{T_j\}_j \subset \mathcal{L}(\mathcal{H})$ is w/ respect to norm. I.e.,

  T_j converges to T w.r.t. <u>norm topology</u> if $\leftarrow$ (or <u>uniform topology</u>)

  $$\lim_{j\to\infty} \|T_j - T\| = 0$$

- However, in quantum physical applications, this notion of convergence is too strong.

- So then we consider other notions of convergence that are more compatible w/ physical intuition.

- One such notion is called <u>weak convergence</u>, or <u>convergence in the weak operator topology</u>:

<u>Def:</u> A sequence $\{T_j\}_j \subset \mathcal{L}(\mathcal{H})$ converges to $T \in \mathcal{L}(\mathcal{H})$ <u>weakly</u> (or in weak op. topology) if

$$\forall \psi, \varphi \in \mathcal{H} \quad \lim_{j\to\infty} \left| \langle \varphi | T_j \psi \rangle - \langle \varphi | T \psi \rangle \right| = 0.$$

Prop. If a sequence $\{T_j\}_j \subset \mathcal{L}(H)$ converges to $T \in \mathcal{L}(H)$ in norm, then it also converges to T weakly.

Proof: $\forall \psi, \varphi \in H$, we have that

$$|\langle \varphi | T_j \psi \rangle - \langle \varphi | T \psi \rangle|$$
$$= |\langle \varphi | (T_j - T) | \psi \rangle|$$
$$\leq \|\varphi\| \, \|\psi\| \, \|T_j - T\|$$

Taking $\lim_{j \to \infty}$, we conclude.

Example: Let $\{\Pi_j\}_j$ be a sequence of orthogonal projections

Let $\{\varphi_j\}_{j=1}^{\infty}$ be an O.N. basis

Then Π_j is projection onto

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$\text{span}\{\varphi_k : k \in \{1, \dots, j\}\}$.

Then consider that

$$|\langle \phi | \pi_j \psi \rangle - \langle \phi | \psi \rangle|$$

$$= |\langle \phi | (I - \pi_j) | \psi \rangle|$$

Now write $|\psi\rangle$ as

$$|\psi\rangle = \sum_{j=1}^{\infty} \alpha_j |\varphi_j\rangle$$

Then $(I - \pi_j)|\psi\rangle = \sum_{\ell=j}^{\infty} \alpha_\ell |\varphi_\ell\rangle$

so that

$$= |\langle \phi | \sum_{\ell=j}^{\infty} \alpha_\ell |\varphi_\ell\rangle|$$

$$\leq \|\phi\| \sum_{\ell=j}^{\infty} |\alpha_\ell|^2 \qquad (C\text{-}S)$$

Then $\lim_{j \to \infty} \sum_{\ell=j}^{\infty} |\alpha_\ell|^2 = 0$

However, what about convergence in norm topology?

Fix j. Consider that

$$\|I - \Pi_j\| = 1$$

(by picking some unit vector in space spanned by $I - \Pi_j$)

then

$$\lim_{j \to \infty} \|I - \Pi_j\| = 1$$

& there is no convergence

This happens b/c order of optimizations is different than in weak topology.

In quantum optics, one could have projections onto photon-number subspace of photon number $\leq N$.
Even though it is intuitive that these should converge to identity, it doesn't happen in norm topology.

Another example: (related)

Consider a sequence $\{T_j\}_j \subset \mathcal{L}(\mathcal{H})$ of shift operators $T_j = A^j$, where A is forward shift operator.

Then

$$T_j(\xi_0, \xi_1, \ldots) = (\underbrace{0, \ldots, 0}_{j \text{ times}}, \xi_0, \xi_1, \ldots)$$

Then $\lim\limits_{j \to \infty} T_j = 0$ in weak O.T.

Then for $\xi, \eta \in \ell^2(\mathbb{N})$, we find that

$$|\langle \eta | T_j \xi \rangle|$$

$$= \left| \sum_{k=0}^{\infty} \bar{\eta}_{k+j} \xi_k \right| \leq \|\xi\| \sum_{k=j}^{\infty} |\eta_k|^2$$

Taking limit $j \to \infty$, RHS $\to 0$

$\Rightarrow T_j \xrightarrow{\text{weakly}} 0$.

However $\|T_j g\| = \|g\| \quad \forall j$

& so there is no convergence to 0 in operator norm topology.

In fact, we can conclude that $\{T_j\}_j$ does not converge in norm topology.

Since $\{T_j\}_j$ converges weakly to 0, ~~norm~~ it does not converge weakly to any other operator.

contrapositive of earlier prop is:

If $\{T_j\}$ does not converge weakly to T, then it does not converge in norm to T.

So $\{T_j\}$ does not converge to any operator T in norm.

Whenever we write equalities for operators $A, B \in \mathcal{L}(\mathcal{H})$

$$A = B,$$

this should be understood in weak sense

$A = B$ means $\langle \varphi | A \psi \rangle = \langle \varphi | B \psi \rangle$ $\forall \varphi, \psi \in \mathcal{H}$.

E.g. $I = \sum_{j=1}^{\infty} |e_j\rangle\langle e_j|$ in weak sense.

- $T \in \mathcal{L}(\mathcal{H})$ is normal if $TT^\dagger = T^\dagger T$
- This holds for selfadjoint + unitary operators

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For such operators ~~the~~ (bounded) that are also compact, (for any bounded sequence $\{x_n\}_n$, the seq. $\{Tx_n\}_n$ has a convergent subsequence.) there is a spectral decomposition.

I.e., $\exists$ a sequence $\{r_j\}_j$ of complex numbers & an O.N. basis $\{\varphi_j\}_j$ such that

$$T = \sum_{j=1}^{\infty} r_j |\varphi_j\rangle\langle\varphi_j|$$

This means that action on $|\psi\rangle$ is

$$T|\psi\rangle = \sum_{j=1}^{\infty} r_j \langle\varphi_j|\psi\rangle |\varphi_j\rangle$$

(any trace-class op. is compact.)

More generally, any compact operator T can be written as

$$T = \sum_{j=1}^{\infty} s_j |\varphi_j\rangle\langle\phi_j|$$

for non-negative s_j & O.N. bases $\{|\varphi_j\rangle\}_j$ $\{|\phi_j\rangle\}_j$

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Duality of bounded operators & trace class operators

- A linear mapping f from complex vector space V to $\mathbb{C}$ is called a linear functional.

- If V is normed, then V^* denotes the set of all continuous linear functionals, called dual space of V.

- can define a norm on V^* by

$$\|f\| = \sup_{\|v\|=1} |f(v)|$$

Important theorem:

Riesz representation theorem

Let $f \in H^*$. Then there exists a unique vector $\phi \in H$ s.t.

$$f(\psi) = \langle \phi | \psi \rangle .$$

Also, $\|f\| = \|\phi\|$

This means that there is only one way, through inner product, to realize the continuous linear functionals.

This then extends to bounded operators + trace-class operators.

What is the dual space $\mathcal{T}(\mathcal{H})^*$ of trace-class operators?

For each $S \in \mathcal{L}(\mathcal{H})$, define linear functional f_S on $\mathcal{T}(\mathcal{H})$ by

$$f_S(T) = \mathrm{tr}\,[ST]$$

Theorem: The mapping $S \to f_S$ is a linear bijection from $\mathcal{L}(\mathcal{H})$ to $\mathcal{T}(\mathcal{H})^*$ & $\|S\| = \|f_S\|$ $\forall S \in \mathcal{L}(\mathcal{H})$.

can identify dual space $\mathcal{T}(\mathcal{H})^*$ w/ $\mathcal{L}(\mathcal{H})$.

can conclude that

a) $S \geq 0 \iff f_S(T) \geq 0 \;\forall\, T \geq 0$.

b) $S = S^\dagger \iff f_S(T) \in \mathbb{R} \;\forall\, T = T^\dagger$.

Quantum Mechanics

$$S(\mathcal{H}) = \{\rho \in \mathcal{T}(\mathcal{H}) : \rho \geq 0 \text{ \& } \mathrm{Tr}[\rho] = 1\}$$

Theorem: State $\rho \in S(\mathcal{H})$ has canonical convex decomposition of the form

$$\rho = \sum_j f_j P_j$$

where $\{f_j\}_j$ is finite or ∞ sequence of ~~non-negative~~ positive numbers $\sum$ to 1,

& $\{P_j\}_j$ is a set of orthogonal projections.

Effect is a mapping from the set $S(\mathcal{H})$ of states to interval $[0,1]$.

i.e. $\rho \to E(\rho) \in [0,1]$.

$E(\rho)$ is the probability of a "yes" answer to "the recorded measurement outcome belongs to a subset $X \subseteq \Omega$."

Basic assumption is that

$$E(\lambda \rho_1 + (1-\lambda)\rho_2) = \lambda E(\rho_1) + (1-\lambda) E(\rho_2)$$

$$\forall \rho_1, \rho_2 \in S(\mathcal{H}) \ \& \ \lambda \in [0,1].$$

affine mapping from $S(\mathcal{H})$ to $[0,1]$.

Prop.: Let E be an effect.

then there exists $\hat{E} \in \mathcal{L}_s(\mathcal{H})$

such that

$$E(\rho) = \mathrm{Tr}[\hat{E}\rho] \quad \forall \rho \in S(\mathcal{H}).$$

Also $0 \leq \hat{E} \leq I$.

Proof: Extend E to a continuous linear functional $\tilde{E}$ on $\mathcal{T}(\mathcal{H})$.

then apply duality $\mathcal{T}(\mathcal{H})^* = \mathcal{L}(\mathcal{H})$.

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Partial trace:

$$\mathrm{Tr}_A : \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{T}(\mathcal{H}_B)$$

is linear mapping satisfying

$$\mathrm{Tr}\left[\mathrm{Tr}_A[T_{AB}] E_B \right] = \mathrm{Tr}\left[T_{AB} (I_A \otimes E_B) \right]$$

$\forall T_{AB} \in \mathcal{T}(\mathcal{H}_A \otimes \mathcal{H}_B)$ & $E_B \in \mathcal{L}(\mathcal{H}_B)$.

How to calculate partial trace?

pick O.N. bases $\{\psi_j\}_j$ for $\mathcal{H}_A$ & $\{\varphi_k\}_k$ for $\mathcal{H}_B$ & then

$$\mathrm{Tr}_A[T] = \sum_{j,k,n} \langle \psi_j|_A \otimes \langle \varphi_k|_B \, T_{AB} \, |\psi_j\rangle_A \otimes |\varphi_n\rangle_B \; |\varphi_k\rangle\langle\varphi_n|_B$$

this is what is meant by

$$\sum_j \langle \psi_j|_A \otimes I_B \,(T_{AB})\, |\psi_j\rangle_A \otimes I_B .$$

state purification:

- given $\rho_A \in S(\mathcal{H}_A)$, purification is $|\psi\rangle_{RA} \in \mathcal{H}_R \otimes \mathcal{H}_A$

such that

$$\mathrm{Tr}_R\left[|\psi\rangle\langle\psi|_{RA}\right] = \rho_A .$$

- can construct from spectral decomposition $\rho = \sum_j f_j |\varphi_j\rangle\langle\varphi_j|$

$$\rightarrow |\psi\rangle_{RA} = \sum_j \sqrt{f_j}\, |\varphi_j\rangle_R |\varphi_j\rangle_A .$$

Lemma: Let $T \in \mathcal{T}(\mathcal{H})$.

Then $$\|T\|_1 = \sup_{U \in \mathcal{U}(\mathcal{H})} \left|\mathrm{Tr}\{UT\}\right|$$

Proof: Use Hölder inequality & polar decomposition & Russo-Dye theorem

Operations & channels

Def. A linear mapping $\mathcal{N}_{A\to B}: \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$

is completely positive if

if $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}$ is positive

$\forall$ finite-dimensional $\mathcal{H}_R$.

- recall map $\mathcal{M}$ is positive if

$\mathcal{M}(T) \geq 0 \quad \forall\, T \geq 0 \;\&\; T \in \mathcal{T}(\mathcal{H})$.

Def. $\mathcal{N}_{A\to B}: \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$

is a channel if it is CP

& trace preserving

$\mathrm{Tr}\{\mathcal{N}(T)\} = \mathrm{Tr}\{T\} \quad \forall\, T \in \mathcal{T}(\mathcal{H})$.

Adjoint of a linear mapping

$\mathcal{N}$ on $\mathcal{T}(\mathcal{H})$ is $\mathcal{N}^{+}$ on $\mathcal{L}(\mathcal{H})$
and defined as $\mathcal{N}^{+}$ satisfying

$$\mathrm{Tr}[\mathcal{N}(T)\,E] = \mathrm{Tr}\{T\,\mathcal{N}^{+}(E)\}$$

$\forall\, T \in \mathcal{T}(\mathcal{H})$ & $E \in \mathcal{L}(\mathcal{H})$.

then $\mathcal{N} : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$

defines $\mathcal{N}^{+} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$.

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to go from linear mapping on $\mathcal{L}(\mathcal{H})$

to channels, we require CP, unital,

& <u>normal</u>.

see text for this.

Stinespring

If a linear map $\mathcal{N}^\dagger : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$

is CP, unital, & normal, then

$\exists\, \mathcal{H}_E$, $\quad V \in \mathcal{L}(\cancel{\mathcal{H}_B \otimes \mathcal{H}_E, \mathcal{H}_A})$

$\mathcal{H}_A, \mathcal{H}_B \otimes \mathcal{H}_E$

such that

$$\mathcal{N}^\dagger(T_B) = V^\dagger (T_B \otimes I_E) V$$

w/ $V^\dagger V = I_A$. $\quad \forall\, T_B \in \mathcal{L}(\mathcal{H}_B)$.

In Schrödinger picture,

for channel $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$

$\exists\, \mathcal{H}_E$, isometry $V \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B \otimes \mathcal{H}_E)$

s.t.

$$\mathcal{N}(X_A) = \mathrm{Tr}_E[V X_A V^\dagger] \quad \forall\, X_A \in \mathcal{T}(\mathcal{H}_A)$$

Operator-sum form

Prop: $\mathcal{N} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ is a channel iff $\exists$ sequence of bounded op's $\{A_k\}_k$ s.t.

$$\mathcal{N}(T) = \sum_k A_k T A_k^\dagger \ , \quad \sum_k A_k^\dagger A_k = I$$

$$\forall\, T \in \mathcal{T}(\mathcal{H}_A).$$