Lecture 3

Continuing w/ bounded operators:

$\sqrt{\cdot}$-lemma: Let $T \in \mathcal{L}_s(H)$ be PSD.

There is a unique PSD operator $\sqrt{T}$ satisfying $(\sqrt{T})^2 = T$

How to prove this?

T is s.a. & spectrum $\sigma(T) \subset \mathbb{R}^+$

- the continuous functional calculus applies to T.

What is continuous functional calculus?

- can get any polynomial function $p(\cdot)$ of T in usual way as

$$p(T)$$

- Stone-Weierstrauss theorem concerning uniform convergence of polynomial functions to arbitrary continuous functions on an interval then applies
- $\sigma(T)$ is compact, contained in interval $[-\|T\|, \|T\|]$
  $\rightarrow$ for PSD $[0, \|T\|]$
- the function $x \to \sqrt{x}$ is continuous on $\sigma(T) \subset \mathbb{R}^+$

can then define $\sqrt{T}$ in this way.

can prove that this is also unique.

can use square-root to define absolute value of bounded operator:

Let $T \in \mathcal{L}(\mathcal{H})$. Abs. val. of T is $|T| \equiv \sqrt{T^\dagger T}$

Can use this to get polar decomposition of an arbitrary bounded operator:

Polar decomp: Let $T \in \mathcal{L}(\mathcal{H})$, then there exists $V \in \mathcal{L}(\mathcal{H})$ such that

$$T = V\,|T|$$

where $\underbrace{\|V\psi\| = \|\psi\|}_{V \text{ is partial isometry.}}$ $\forall \psi \in \mathrm{supp}(V)$

(can have a kernel)

Proof: $\forall\ \psi, \phi \in \mathcal{H}$

$$\langle |T|\psi \,|\, |T|\phi \rangle = \langle \psi \,|\, |T|^2 \phi \rangle$$
$$= \langle \psi |\, T^{+} T \phi \rangle$$
$$= \langle T\psi \,|\, T\phi \rangle$$

this implies that the mapping

$$|T|\phi \rightarrow T\phi$$

is isometric. this map is V.

Note that it is from $\mathrm{ran}(|T|)$ to $\mathrm{ran}(T)$. It is a partial isometry by setting it to 0 for all vectors in $\ker(|T|)$.

Every ~~s.a.~~ $T \in \mathcal{L}_s(\mathcal{H})$ can be written as

$$T = T^+ - T^-$$

where
$$T^+ = \frac{1}{2}(|T| + T)$$
$$T^- = \frac{1}{2}(|T| - T)$$

where $T^+, T^- \geq 0$ &
$$T^+ T^- = 0$$

Unitary operators

Let U be a linear map on $\mathcal{H}$.

T.F.A.E.:

1) U is an isomorphism
2) U is an "onto" isometry
3) U is bounded & $UU^+ = U^+U = I$

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Recall isomorphism U is s.t.

$$\langle U\varphi | U\psi\rangle = \langle \varphi|\psi\rangle \quad \forall \varphi,\psi \in \mathcal{H}.$$

1) $\Rightarrow$ 2). Follows by picking $\varphi=\psi$ so that $\|U\psi\| = \|\psi\| \ \forall \psi \in \mathcal{H}$.

2) $\Rightarrow$ 3) Suppose U is onto isometry

$\Rightarrow$ U is bounded w/ $\|U\|=1$

Also conclude that U is one-to-one

$\forall \psi,\varphi \in \mathcal{H}$ $\quad \|U\psi - U\varphi\| = \|\psi-\varphi\|$

$\Rightarrow$ $U\psi = U\varphi$ iff $\psi = \varphi$

$\Rightarrow$ U is bijective & has an inverse.
(both one-to-one & onto)

Also $\langle\varphi|\varphi\rangle = \langle U\varphi|U\varphi\rangle$
$= \langle \varphi | U^{+}U\varphi\rangle$
$\forall \varphi \in \mathcal{H} \Rightarrow U^{+}U = I.$

$\Rightarrow U^{-1} = U^+$ Also $\Rightarrow$ $UU^+ = I$

3) $\Rightarrow$ 1) Suppose $U \in \mathcal{L}(\mathcal{H})$ +

$$UU^+ = U^+U = I.$$

Then U is bijective w/ $U^{-1} = U^+$

+ $\forall \varphi, \psi \in \mathcal{H}$

$$\langle \varphi | \psi \rangle = \langle \varphi | U^+ U \psi \rangle$$
$$= \langle U\varphi | U\psi \rangle$$

$\Rightarrow$ U is isomorphism

Def.: $U \in \mathcal{L}(\mathcal{H})$ is unitary if

$$UU^+ = U^+U = I.$$

Need both conditions in inf. dim. space

Remember shift op. A satisfies

$$A^+A = I \text{ but } AA^+ \neq I$$

b/c $AA^+ \delta_0 = 0$.

All eigenvalues of $\overset{\text{unitary}}{\wedge}$ U satisfy

$$|\lambda| = 1$$

Suppose $U\psi = \lambda\psi$ for some $\psi \in \mathcal{H}$.
non-zero

then $\langle\psi|\lambda^*\lambda|\psi\rangle = \langle\psi|U^+U|\psi\rangle$

$= |\lambda|^2\langle\psi|\psi\rangle$ $= \langle\psi|\psi\rangle$

$$\Rightarrow |\lambda| = 1.$$

<u>Connection between unitary + self-adjoint op's.</u>

Consider exponential map on $\mathcal{L}(\mathcal{H})$.

Let $T \in \mathcal{L}(\mathcal{H})$.

For $k \in \mathbb{N}$, define

$$F_k(T) \equiv \sum_{n=0}^{k} \frac{T^n}{n!} \qquad f_k(T) \equiv \sum_{n=0}^{k} \frac{\|T^n\|}{n!}$$

where $T^0 = I$

Then $f_k(T) \leq \sum_{n=0}^{k} \frac{\|T\|^n}{n!} \leq \sum_{n=0}^{\infty} \frac{\|T\|^n}{n!}$

$= e^{\|T\|}$

$< \infty$

$\Rightarrow$ sequence $\{f_k(T)\}_k$ has an upper bound & converges.

$\Rightarrow$ $\sum_{n=0}^{\infty} \frac{T^n}{n!}$ is absolutely convergent

& since $\mathcal{L}(H)$ is complete normed space,

abs. conv. series converges.

Let e^T denote limit of series

$$e^T \equiv \lim_{k\to\infty} F_k(T) .$$

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products & adjoint maps are continuous on $\mathcal{L}(H)$

$$\Rightarrow \quad e^{aT} e^{bT} = e^{(a+b)T},$$

$$\left(e^{aT}\right)^{+} = e^{\bar{a}T^{+}}$$

$\forall a, b \in \mathbb{C}$ & $T \in \mathcal{L}(H)$

For $T \in \mathcal{L}_s(H)$

$$\left(e^{iT}\right)^{+} = e^{-iT}$$

$$\& \quad e^{iT} e^{-iT} = e^{-iT} e^{iT} = e^{0} = I$$

$\Rightarrow e^{iT}$ is unitary for $T \in \mathcal{L}_s(H)$.

Trace-class operators

trace in finite-dim. is just sum of diagonal elements in some basis.

Not as straight forward for ∞-dim. case.

Trace is meaningful for only a proper subset of bounded operators.

- trace should be $< \infty$.
  & independent of o.N. basis.

Let $\mathcal{H}$ be sep. Hilbert space & $\{\varphi_j\}_{j=1}^{\infty}$ an o.N. basis.

For PSD $T \in \mathcal{L}(\mathcal{H})$, write

$$\mathrm{Tr}[T] = \sum_{j=1}^{\infty} \langle \varphi_j | T\varphi_j \rangle .$$

Due to $T \geq 0$, RHS is $\sum$ of ~~pos~~ non-negative numbers.

If $\sum$ does not converge, then $\mathrm{Tr}[T] = \infty$.

Tr[T] does not depend on basis chosen when $T \geq 0$.

Let $\{\psi_j\}_{j=1}^{\infty}$ be another O.N. basis.

Use Parseval's theorem, + $\sqrt{\cdot}$-lemma

$$\sum_j \langle \psi_j | T \psi_j \rangle = \sum_j \| T^{1/2} \psi_j \|^2$$

$$= \sum_j \sum_k |\langle \varphi_k | T^{1/2} \psi_j \rangle|^2$$

$$= \sum_k \sum_j |\langle \psi_j | T^{1/2} \varphi_k \rangle|^2$$

$$= \sum_k \| T^{1/2} \varphi_k \|^2$$

$$= \sum_k \langle \varphi_k | T \varphi_k \rangle$$

exchange of sum is OK by Tonelli's theorem

(all numbers are non-negative + so sum can go in either order)

<u>Def.</u> A bounded operator T is trace-class if $\mathrm{Tr}[|T|] < \infty$.

Denote trace class operators by

$$\mathcal{T}(\mathcal{H}).$$

<u>Examples</u> I is bounded but is not trace class

shift A is bounded ($\|A\|=1$) but is not trace class

$$|A| = (A^\dagger A)^{1/2} = I^{1/2} = I$$

Note that

$$\sum_{j=0}^{\infty} \langle \delta_j | A \delta_j \rangle = \sum_{j=0}^{\infty} \langle \delta_j | \delta_{j+1} \rangle = 0.$$

So trace is tricky in ∞-dim + follow def's.

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<u>Prop:</u> If $T \in \mathcal{T}(\mathcal{H})$ & $\{\varphi_j\}_{j=1}^{\infty}$ is an O.N. basis, then

$$Tr[T] = \sum_{j=1}^{\infty} \langle \varphi_j | T \varphi_j \rangle$$

is trace of T & is independent of basis chosen.

<u>Proof:</u> Define bounded operator T to be compact if

$$\lim_{j \to \infty} \| T \varphi_j \| = 0$$

$\forall$ O.N. bases $\{\varphi_j\}_{j=1}^{\infty}$.

can then write T as

$$\sum_{j=1}^{\infty} \lambda_j \, |\psi_j\rangle\langle\phi_j|$$ (NOT PROVEN)

for sequence $\{\lambda_j\}_j \in \mathbb{R}^+ \setminus \{0\}$ being finite or converging to 0.

$\{\psi_j\}_j$ + $\{\phi_j\}$ O.N. families positive trace-class operator T is compact

b/c

$$\sum_j \|\sqrt{T}\varphi_j\|^2 = \sum_j \langle \varphi_j | T \varphi_j \rangle = \mathrm{Tr}[T] < \infty$$

$\Rightarrow \|\sqrt{T}\varphi_j\| \to 0$ as $j \to \infty$.

$\Rightarrow \sqrt{T}$ is compact

$\Rightarrow T$ is compact

For arbitrary trace-class T,

$|T|$ is trace-class & so

$|T|$ is compact.

By polar decomposition,

$T = U|T|$ & so T is compact.

Now write

$$T = \sum_j s_j |\psi_j\rangle\langle\phi_j|$$

$$|\mathrm{Tr}\{T\}| = \left|\sum_k \langle \varphi_k | T \varphi_k \rangle\right|$$

$$\leq \sum_k |\langle \varphi_k | T \varphi_k \rangle|$$

$$\leq \sum_{k,j} \lambda_j \, |\langle \varphi_k | \psi_j \rangle| \, |\langle \phi_j | \varphi_k \rangle|$$

$$\leq \sum_j \lambda_j \left[\sum_k |\langle \varphi_k | \psi_j \rangle|^2\right]^{1/2} \cdot \left[\sum_k |\langle \varphi_k | \phi_j \rangle|^2\right]^{1/2}$$

$$= \cancel{\sum_j \lambda_j = \mathrm{Tr}[|T|] < \infty}$$

$$= \sum_j \lambda_j \, \|\psi_j\| \, \|\phi_j\|$$

$$= \sum_j \lambda_j = \mathrm{Tr}[|T|] < \infty .$$

establishes absolute convergence

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By abs. convergence & Fubini's theorem (regarding exchanges of sums

$$\sum_k \langle \varphi_k | T \varphi_k \rangle = \sum_k \sum_j \lambda_j \langle \varphi_k | \psi_j \rangle \langle \phi_j | \varphi_k \rangle$$

$$= \sum_j \lambda_j \sum_k \langle \phi_j | \varphi_k \rangle \langle \varphi_k | \psi_j \rangle$$

$$= \underbrace{\sum_j \lambda_j \langle \phi_j | \psi_j \rangle}$$

does not depend on choice of $\{|\varphi_k\rangle\}_k$

Define trace norm by

$$T \to \|T\|_1 \equiv \mathrm{Tr}[|T|].$$

operator norm is bounded by trace norm

$$\|T\| \leq \|T\|_1 \quad \forall\, T \in \mathcal{T}(\mathcal{H}).$$

Even though $\mathrm{Tr}[S]$ is not defined $\forall$ bounded S

$\mathrm{Tr}[ST]$ is defined whenever $T \in \mathcal{T}(\mathcal{H})$.

Also

$$\mathrm{Tr}[ST] = \mathrm{Tr}[TS]$$

$$\& \ |\mathrm{Tr}[ST]| \leq \|T\|_1 \|S\|.$$

Hilbert-Schmidt operators are those for which

$$\|T\|_2 \equiv \|T\|_{HS} \equiv \mathrm{Tr}\left[T^\dagger T\right]^{1/2} < \infty.$$

$$\|T\| \leq \|T\|_2 \leq \|T\|_1$$

Cauchy-Schwarz

$$\left|\mathrm{Tr}\left[ST\right]\right|^2 \leq \|S\|_2^2 \, \|T\|_2^2 .$$