# Lecture 2

Operators on Hilbert spaces

Mapping $T: \mathcal{H} \to \mathcal{H}$ is linear if

$$T(c\psi + \phi) = cT(\psi) + T(\phi)$$

$$\forall \psi, \phi \in \mathcal{H} \ \& \ c \in \mathbb{C}$$

Abbreviate $T(\psi) = T\psi$

Linear mapping is a <u>bounded</u> operator if $\exists\ t \geq 0$ such that

$$\|T\psi\| \leq t\|\psi\| \qquad \forall \psi \in \mathcal{H}$$

That is, $T\psi \in \mathcal{H}$ if $\psi \in \mathcal{H}$.
Use $\mathcal{L}(\mathcal{H})$ to denote set of bounded operators.

<u>Example</u>: photon number operator

$$\hat{n} = \sum_{n=0}^{\infty} n|n\rangle\langle n| \quad \text{is unbounded.}$$

Pick Basel state

$$|\beta\rangle = \sqrt{\frac{6}{\pi^2}} \sum_{n=1}^{\infty} \sqrt{\frac{1}{n^2}}\, |n\rangle$$

then

$$\langle\beta|\hat{n}|\beta\rangle = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Another example: (more abstract)

For each $f \in \ell^2(\mathbb{N})$,
define $Nf : \mathbb{N} \to \mathbb{C}$ by

$$(Nf)(n) = n f(n)$$

Then $Nf$ is not in $\ell^2(\mathbb{N})$.

E.g., take $f(n) = \begin{cases} 0 & \text{if } n=0 \\ \frac{1}{n} & \text{if } n>0. \end{cases}$

Then $f \in \ell^2(\mathbb{N})$ but $Nf \notin \ell^2(\mathbb{N})$.

Various subspaces related to a bounded operator $T$:

(kernel) $\ker(T) = \{\psi \in \mathcal{H} : T\psi = 0\}$

(range) $\mathrm{ran}(T) = \{\psi \in \mathcal{H} : \psi = T\varphi$ for some $\varphi \in \mathcal{H}\}$

(support) $\mathrm{supp}(T) = \{\psi \in \mathcal{H} : \psi \perp \varphi$ for all $\varphi \in \ker(\mathcal{H})\}$

dimension of $\mathrm{supp}(T)$ is called rank of $T$.

Set of bounded operators is a vector space.

I.e.,

$$(S+T)\psi = S\psi + T\psi$$
$$(cT)\psi = c(T\psi)$$

Vector space $\mathcal{L}(H)$ (bounded operators)

is a normed space, w/ norm defined via

$$\|T\| = \sup_{\|\psi\|=1} \|T\psi\|$$

Also, $\|T\|$ is the least number $t$ satisfying

$$\|T\psi\| \leq t\|\psi\| \quad \forall \psi \in H.$$

called operator norm or spectral norm

- can show that $\mathcal{L}(H)$ is complete in the operator norm topology.

- follows directly from the definition of operator norm that if $T \in \mathcal{L}(H)$ then $\forall \psi \in H$, we have

$$\|T\psi\| \leq \|T\| \, \|\psi\|$$

Combining w/ Cauchy-Schwarz gives that $\forall \varphi, \psi \in \mathcal{H}$

$$|\langle \varphi | T\psi \rangle| \leq \|\varphi\| \ \|\psi\| \ \|T\|$$

- If $S, T \in \mathcal{L}(\mathcal{H})$, then

$$\|ST\psi\| \leq \|S\| \ \|T\psi\| \leq \|S\| \ \|T\| \ \|\psi\|$$

$$\Rightarrow \quad \|ST\| \leq \|S\| \ \|T\|$$

$\Rightarrow \quad ST \in \mathcal{L}(\mathcal{H})$ (ST is bounded)

$\Rightarrow \mathcal{L}(\mathcal{H})$ is an algebra.

- Define the adjoint $T^\dagger$ of $T \in \mathcal{L}(\mathcal{H})$ by

$$\langle \varphi | T^\dagger \psi \rangle = \langle T\varphi | \psi \rangle$$

$$\forall \psi, \varphi \in \mathcal{H}.$$

Mapping $T \to T^{\dagger}$ is conjugate linear:

$$(cT+S)^{\dagger} = \bar{c}\, T^{\dagger} + S^{\dagger}$$

$\forall\, T, S \in \mathcal{L}(\mathcal{H})$ & $c \in \mathbb{C}$

&

$$(ST)^{\dagger} = T^{\dagger} S^{\dagger}$$

$$(T^{\dagger})^{\dagger} = T$$

<u>Prop.</u>: A bounded operator $T$ & its adjoint $T^{\dagger}$ satisfy

$$\|T\| = \|T^{\dagger}\| = \|T^{\dagger}T\|^{1/2}$$

<u>Proof</u>: $\forall\, \psi \in \mathcal{H}$ s.t. $\|\psi\| = 1$

$$\|T\psi\|^2 = |\langle T\psi | T\psi \rangle|$$

$$= |\langle \psi | T^{\dagger}T\psi \rangle|$$

$$\leq \|\psi\|^2 \, \|T^{\dagger}T\| = \|T^{\dagger}T\|$$

$$\leq \|T^+\| \, \|T\|$$

$$\Rightarrow \quad \|T\|^2 \leq \|T^+T\| \leq \|T^+\| \, \|T\| \quad (*)$$

$$\Rightarrow \quad \|T\| \leq \|T^+\|$$ (by dividing by $\|T\|$)

Repeating the above w/

$$T \to T^+$$
$$T^+ \to T$$

+ using $(T^+)^+ = T$ gives that

$$\|T^+\| \leq \|T\|$$

$$\Rightarrow \|T\| = \|T^+\|$$

$$+ \ (*) \Rightarrow \quad = \|T^+T\|^{1/2}$$

All of these properties imply that $\mathcal{L}(\mathcal{H})$ is a $C^*$-algebra.

- $\mathcal{L}(\mathcal{H})$ is an algebra
- $\mathcal{L}(\mathcal{H})$ is a complete normed space
- adjoint mapping $T \to T^+$ on $\mathcal{L}(\mathcal{H})$ is conjugate linear & satisfies $(ST)^+ = T^+ S^+$, $(T^+)^+ = T$
- operator norm satisfies

$$\|ST\| \leq \|S\| \, \|T\|$$

$$\|T\| = \|T^+\| = \|T^+ T\|^{1/2}$$

<u>Example</u>: Shift operator

write $g \in \ell^2(\mathbb{N})$ as

$$g = (g_0, g_1, \dots)$$

where $g_i = g(i)$.

Define forward shift operator $A$

as $$A(g_0, g_1, \dots) = (0, g_0, g_1, \dots)$$

$$\Rightarrow \|Ag\| = \sum_{j=0}^{\infty} |g_j|^2 = \|g\|$$

& so $A$ is bounded & $\|A\| = 1$.

Adjoint $A^+$ act as

$$A^+(g_0, g_1, g_2, \dots) = (g_1, g_2, \dots)$$

backward shift operator.

In a finite $d$-dimensional Hilbert space $\mathcal{H}$, set $\mathcal{L}(\mathcal{H})$ consists of all linear mappings & is identified w/ $M_d(\mathbb{C})$, $d \times d$ complex matrices.

- Fix an O.N. basis $\{\varphi_j\}_{j=1}^{d}$ & for each $T \in \mathcal{L}(\mathcal{H})$, define

$$[T]_{j,k} = \langle \varphi_j | T\varphi_k \rangle$$

Conversely, by starting w/ $[T]_{jk}$ we recover action of $T$ on $\psi \in \mathcal{H}$ via

$$T\psi = \sum_{j,k} [T]_{j,k} \langle \varphi_k | \psi \rangle \varphi_j$$

Then $[T^{+}]_{j,k} = \overline{[T]_{k,j}}$

- If $\mathcal{H}$ is $\infty$-dim., then we can fix an O.N. basis for $\mathcal{H}$ & define $[T]_{j,k}$ by

$$\langle \varphi_j | T \varphi_k \rangle$$

- However, a given infinite matrix may not be bounded & it can be difficult to tell.

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Prop.: Let $S, T \in \mathcal{L}(\mathcal{H})$. If $\langle \psi | S\psi \rangle = \langle \psi | T\psi \rangle \;\; \forall \; \psi \in \mathcal{H}$ then $S = T$.

Proof: Suppose that $\langle \psi | S\psi \rangle = \langle \psi | T\psi \rangle$ $\forall \; \psi \in \mathcal{H}$. By polarization identity,

$$\langle \phi | T\psi \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \psi + i^k \phi | T(\psi + i^k \phi) \rangle$$

$\Rightarrow \quad \langle \phi | S\psi \rangle = \langle \phi | T\psi \rangle \quad \forall\ \psi, \phi \in \mathcal{H}.$

Choose $\phi = (S-T)\psi$

+ get that $\langle (S-T)\psi \,|\, (S-T)\psi \rangle = 0$

$\forall\ \psi \in \mathcal{H}$

$\Rightarrow \quad S\psi = T\psi \quad \forall\ \psi \in \mathcal{H}.$

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Eigenvalues for bounded operators:

Let $T$ be a bounded operator.

$\lambda \in \mathbb{C}$ is

1) an eigenvalue of $T$ if

$\exists$ non-zero $\psi \in \mathcal{H}$ such that

$T\psi = \lambda\psi$

$\psi$ is an eigenvector w/eigenvalue $\lambda$.

2) in the spectrum of $T$ if

inverse of $\lambda \cancel{I} - T$ does not exist.

In finite dimensions, every operator has eigenvalues (solutions of $\det(T-\lambda I)=0$).

In $\infty$-dim., an operator need not have eigenvalues

<u>Example</u>: Shift operator does not have eigenvalues.

Suppose $\psi$ is an eigenvector w/ eigenvalue $\lambda$. Write $\psi = \sum_{k=0}^{\infty} c_k \delta_k$

Since $A\delta_k = \delta_{k+1}$

$\Rightarrow$ $A\psi = \lambda\psi$

$$= \sum_{k=0}^{\infty} c_k A\delta_k = \lambda \sum_{k=0}^{\infty} c_k \delta_k$$

$$= \sum_{k=0}^{\infty} c_k \delta_{k+1} = \sum_{k=0}^{\infty} \lambda c_k \delta_k$$

$$= \sum_{k=1}^{\infty} c_{k-1} \delta_k = \sum_{k=0}^{\infty} \lambda c_k \delta_k$$

$\Rightarrow \quad \lambda c_0 = 0, \quad \lambda c_1 = c_0, \quad \lambda c_2 = c_1$

can only happen if $c_0 = c_1 = \cdots = 0$

$\Rightarrow \psi = 0$

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Even though a bounded operator need not have eigenvalues, every non-zero bounded operator has a non-empty spectrum that is bounded by operator norm

- e.g., for shift operator,

0 is in spectrum because $A - 0 = A$ is not invertible (any vector w/ non-zero 1st component is not in its range.)

## Partial order of self-adjoint operators

- bounded operator $T \in \mathcal{L}(\mathcal{H})$ is <u>self-adjoint</u> if $T = T^\dagger$.
  - denote by $\mathcal{L}_s(\mathcal{H})$ the set of bounded self-adjoint operators

$\mathcal{L}_s(\mathcal{H})$ is a real vector space b/c real linear combinations of s.a. op's remain s.a.

complex linear combinations of s.a. op's "fill up" $\mathcal{L}(\mathcal{H})$.

Let $T \in \mathcal{L}(\mathcal{H})$.

Then $T = T_R + i T_I$

where $T_R = \frac{1}{2}(T + T^\dagger)$, $T_I = \frac{T - T^\dagger}{2i}$

Bounded operator is s.a. iff

$T_I = 0$.

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<u>Prop.</u>: $T \in \mathcal{L}(\mathcal{H})$ is s.a. iff

$\langle \psi | T\psi \rangle \in \mathbb{R} \quad \forall \psi \in \mathcal{H}$.

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<u>Proof.</u> By previous, $T \in \mathcal{L}(\mathcal{H})$ is s.a.

iff $\langle \psi | T\psi \rangle = \langle \psi | T^\dagger \psi \rangle \; \forall \psi \in \mathcal{H}$.

But $\langle \psi | T^\dagger \psi \rangle = \langle T\psi | \psi \rangle$, so

$\Rightarrow \quad \langle \psi | T\psi \rangle = \overline{\langle \psi | T\psi \rangle}$

$\forall \psi \in \mathcal{H}$.

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<u>Def.</u> $T \in \mathcal{L}(\mathcal{H})$ is positive semi-definite if $\langle \psi | T\psi \rangle \geq 0$

$\forall \psi \in \mathcal{H}$.

By definition, PSD op's are s.a.