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Lecture 1

Intro to course

Review syllabus

Separable Hilbert spaces

- review background for ∞-dimensional spaces
- Hilbert space is closest possible generalization of finite-dimensional space $\mathbb{C}^d$.
- Hilbert space is a complex inner product space, which is a complete metric space wrt distance function induced by inner product.

Inner product space

Let $\mathcal{H}$ be a complex vector space. Function $\langle \cdot | \cdot \rangle$ on $\mathcal{H} \times \mathcal{H}$ is an inner product if it satisfies the following $\forall\, |\varphi\rangle, |\psi\rangle, |\phi\rangle \in \mathcal{H}$ & $c \in \mathbb{C}$:

1) $\langle \varphi | c\psi + \phi \rangle = c\langle \varphi | \psi \rangle + \langle \varphi | \phi \rangle$

complex conjugate

2) $\overline{\langle \varphi | \psi \rangle} = \langle \psi | \varphi \rangle$

3) $\langle \psi | \psi \rangle \geq 0$ &

$\langle \psi | \psi \rangle = 0$ iff $|\psi\rangle = 0$

1) & 2) imply that

$\langle c\varphi + \phi | \psi \rangle = \bar{c}\langle \varphi | \psi \rangle + \langle \phi | \psi \rangle$

Two inner product spaces $\mathcal{H}$ & $\mathcal{H}'$ are <u>isomorphic</u> if $\exists$ a bijective linear mapping $U : \mathcal{H} \to \mathcal{H}'$ such that

$$\langle U\varphi | U\psi \rangle = \langle \varphi | \psi \rangle$$

$\forall\, \varphi, \psi \in \mathcal{H}$. U is called an <u>isomorphism</u>.

- φ & ψ are orthogonal if $\langle \varphi | \psi \rangle = 0$.

- For an inner-product space $\mathcal{H}$, if for any positive d, there exists an orthogonal set of d vectors, then $\mathcal{H}$ is infinite-dimensional. Otherwise, $\mathcal{H}$ is finite-dim.

<u>Example</u>: Inner product space $\ell^2(\mathbb{N})$

Let $\mathbb{N}$ be the set of natural numbers, including 0.

Let $\ell^2(\mathbb{N})$ be a set of functions $f: \mathbb{N} \to \mathbb{C}$ such that

$$\sum_{j=0}^{\infty} |f(j)|^2 < \infty .$$

The formula

$$\langle f | g \rangle = \sum_{j=0}^{\infty} \overline{f(j)}\, g(j)$$

defines an inner product on $\ell^2(\mathbb{N})$.

We can use Cauchy-Schwarz

$$\left| \sum_{j=0}^{\infty} \overline{f(j)}\, g(j) \right|^2 \leq \left[\sum_{j=0}^{\infty} |f(j)|^2 \right] \left[\sum_{k=0}^{\infty} |g(k)|^2 \right]$$

to verify that $\langle f | g \rangle$ is finite when $f, g \in \ell^2(\mathbb{N})$.

For each $k \in \mathbb{N}$, let δ_k denote Kronecker delta, defined as

$$\delta_k(j) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{else} \end{cases}$$

Then $\langle \delta_k | \delta_l \rangle = 0$ if $k \neq l$

& so then $\ell^2(\mathbb{N})$ is ∞-dimensional since collections of Kronecker functions is an orthogonal set.

Every inner product space is a normed space, w/ norm defined as

$$\|\psi\| \equiv \sqrt{\langle\psi|\psi\rangle}$$

Following conditions are then satisfied $\forall\, \psi, \varphi \in \mathcal{H}$ & $c \in \mathbb{C}$

1) $\|\varphi\| \geq 0$ & $\|\varphi\| = 0$ iff $\varphi = 0$

2) $\|c\varphi\| = |c|\,\|\varphi\|$

3) $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$ (follows from Cauchy-Schwarz)

Geometric Intuition still holds:

Pythagoras: $\|\varphi + \psi\|^2 = \|\varphi\|^2 + \|\psi\|^2$ for φ, ψ orthogonal

Parallelogram law:

$$\|\varphi+\psi\|^2 + \|\varphi-\psi\|^2 = 2\|\varphi\|^2 + 2\|\psi\|^2$$

Norm induces a metric on H.

Distance between $\psi, \varphi \in H$ is

$$\|\psi-\varphi\|$$

- can use this to prove that the mapping $\varphi \to \langle\psi|\varphi\rangle$ is continuous.

(From Cauchy - Schwarz

$$|\langle\psi|\varphi_1\rangle - \langle\psi|\varphi_2\rangle|$$

$$= |\langle\psi|\varphi_1-\varphi_2\rangle|$$

$$\leq \|\psi\|\,\|\varphi_1-\varphi_2\|$$

Metric space is called <u>complete</u> if every Cauchy sequence is convergent (limit is also in space)

Recall that a sequence $\{\varphi_j\}_j$ is a Cauchy sequence if $\forall\, \varepsilon > 0$ $\exists\, N_\varepsilon$ such that

$$d(\varphi_j, \varphi_k) \leq \varepsilon \quad \text{whenever}$$
$$j, k \geq N_\varepsilon$$

(Counter) <u>Example</u>: The space of rational numbers, w/ metric given by absolute value of the difference, is not complete.

- Consider the sequence

$$x_1 = 1, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

This is Cauchy, but does not converge to a rational.

If it did have a limit, then it would have to be the solution to

$$x = \frac{x}{2} + \frac{1}{x} \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}$$

but this is not rational

Counterexample: open interval $(0,1)$ w/ standard metric is not complete

sequence $x_n = \frac{1}{n}$ is Cauchy,

but does not have a limit in the space.

- closed interval $[0,1]$ is complete.

> Hilbert space is a complete inner product space.

One useful consequence of completeness is the existence of basis expansions.

An orthogonal set $X \subset H$ is an orthonormal set if each vector $\psi \in X$ has unit norm.

An orthonormal basis for $\mathcal{H}$ is a <u>maximal</u> orthonormal set (no other orthonormal set contains it as a proper subset.)

Maximality of X $\Rightarrow$ if ψ is orthogonal to all vectors in X, then $\psi = 0$.

- Every Hilbert space has an orthonormal basis + all O.N. bases have the same cardinality.

$$\boxed{\text{A Hilbert space is } \underline{\text{separable}} \text{ if it has a countable O.N. basis}}$$

Throughout the course, we will be dealing with & focusing on separable Hilbert spaces.

Example: $\ell^2(\mathbb{N})$ inner product space is complete (can be shown)

The set of Kronecker functions $\{\delta_0, \delta_1, \ldots\}$ is an O.N. basis for $\ell^2(\mathbb{N})$, which follows from criterion for maximality. If $f \in \ell^2(\mathbb{N})$ satisfies $\langle \delta_k | f \rangle = 0 \ \forall k$, then $f(k) = 0$ & so f is identically zero.

Conclusion: $\ell^2(\mathbb{N})$ is a separable Hilbert space.

Any separable Hilbert space is isomorphic to $\ell^2(\mathbb{N})$.

Simple proof sketch: Fix an O.N. basis for separable Hilbert space $\mathcal{H}$ (call it $\{e_k\}_{k=0}^{\infty}$).

For each $\psi \in \mathcal{H}$,
define $\tilde{\psi} : \mathbb{N} \to \mathbb{C}$ by
$\tilde{\psi}(j) = \langle \varphi_j | \psi \rangle$. Then $\tilde{\psi} \in \ell^2(\mathbb{N})$
& $\psi \to \tilde{\psi}$ is an isomorphism
between $\mathcal{H}$ & $\ell^2(\mathbb{N})$.

<u>Counterexample</u>: Is "ideal EPR state" in Hilbert space? Is it a state?
Is it normalizable?

Define

$$|\varphi^{N_s}\rangle_{RA} = \frac{1}{\sqrt{N_s+1}} \sum_{n=0}^{\infty} \sqrt{\left(\frac{N_s}{N_s+1}\right)^n} \, |n\rangle_R |n\rangle_A$$

where $|n\rangle$ are Kronecker functions
or photonic number states.

Can check that

$$\langle \varphi^{N_s} | \varphi^{N_s} \rangle_{RA} = 1$$

so that every $|\varphi^{N_s}\rangle_{RA}$ is normalizable & is thus a legitimate state.

What about "ideal EPR state"?

$$|\varphi^{EPR}\rangle \equiv \lim_{N_s \to \infty} |\varphi^{N_s}\rangle_{RA}$$

Point: the sequence $\{|\varphi^{N_s}\rangle_{RA}\}_{N_s \geq 0}$ is not a Cauchy sequence & so its limit need not be contained in Hilbert space $H_R \otimes H_A$.

Consider that for $N_s, N_s' \geq 0$

$$\| |\varphi^{N_s}\rangle - |\varphi^{N_s'}\rangle \|_2 =$$

$$\sqrt{2 - 2\,\mathrm{Re}\{\langle \varphi^{N_s} | \varphi^{N_s'} \rangle\}}$$

$$= \sqrt{2(1 - \langle \varphi^{N_s} | \varphi^{N_s'} \rangle)}$$

Direct calculation gives that

$$\langle \varphi^{N_s} | \varphi^{N_s'} \rangle = \frac{1}{\sqrt{(N_s+1)(N_s'+1)} - \sqrt{N_s N_s'}}$$

Fix $N_s' \geq 0$, then expand in terms of $N_s = \infty$ to find that

$$\langle \varphi^{N_s} | \varphi^{N_s'} \rangle = \frac{1}{N_s}\left(2N_s' + 1 + 2\sqrt{N_s'(N_s'+1)} \right) + O(1/N_s^2).$$

$\Rightarrow$ for fixed $N_s' \geq 0$,

$$\lim_{N_s \to \infty} \| |\varphi^{N_s}\rangle - |\varphi^{N_s}\rangle \|_2 = 2$$

$\Rightarrow$ $\{ |\varphi^{N_s}\rangle \}_{N_s \geq 0}$ is not Cauchy & so limit is not in the Hilbert space.

Similar argument for position & momentum eigenstates.

Given an O.N. basis, we can write every $\psi \in \mathcal{H}$ as

$$|\psi\rangle = \sum_{k=0}^{\infty} \langle \varphi_k|\psi\rangle |\varphi_k\rangle$$

w/ the meaning of this being that

$$\lim_{N\to\infty} \left\| |\psi\rangle - \sum_{k=0}^{N} \langle \varphi_k|\psi\rangle |\varphi_k\rangle \right\|_2 = 0$$

How to see this?

Consider that

$$\left\| |\psi\rangle - \sum_{k=0}^{N} \langle \varphi_k|\psi\rangle |\varphi_k\rangle \right\|_2^2$$

$$= \langle \psi|\psi\rangle - \sum_{k=0}^{N} |\langle \psi|\varphi_k\rangle|^2$$

$$= \| |\psi\rangle \|^2 - \sum_{k=0}^{N} |\langle \psi|\varphi_k\rangle|^2$$

Now apply Parseval identity to see that limit is 0.