PHYS 7895 Spring 2019
Gaussian Quantum Information
Homework 2

Due Friday, 1 March 2019, by 4pm in Nicholson 447

1. The displacement operator acting on an n-mode state is defined as

$$\hat{D}_r \equiv \exp(i r^T \Omega \hat{r}), \tag{1}$$

where $r \in \mathbb{R}^{2n}$ and $\hat{r}$ is the $2n$-dimensional vector of quadrature operators.

   (a) Prove that $\hat{D}_r$ is a bounded operator.

   (b) Calculate its operator norm.

   (c) What is a unit vector that achieves the operator norm?

   (d) Is $\hat{D}_r$ trace class?

2. Prove that the spectrum of the position-quadrature operator $\hat{x} \equiv \left(\hat{a} + \hat{a}^\dagger\right)/\sqrt{2}$ is equal to the real line. (Recall that the spectrum of an operator M is defined to be the set of all $\lambda \in \mathbb{C}$ such that $M - \lambda I$ is not invertible.)

3. Prove that $\det(\sigma) \geq 1$ and $\sigma > 0$ implies that $\sigma + i\Omega \geq 0$ (uncertainty principle) when σ is the covariance matrix of a single-mode bosonic state.

4. Prove that a 4×4 matrix σ is the covariance matrix of a two-mode bosonic state if and only if $\det(\sigma) - \Delta + 1 \geq 0$, $\Delta^2 \geq 4\det(\sigma)$, and $\sigma > 0$, where Δ is the sum of the squares of the symplectic eigenvalues of σ.

5. Recall that a faithful Gaussian state is defined as

$$\frac{\exp(-\hat{r}^T H \hat{r})}{\mathrm{Tr}[\exp(-\hat{r}^T H \hat{r})]} \tag{2}$$

for H a $2n \times 2n$ real, positive definite matrix H, called the Hamiltonian matrix.

   (a) The single-mode squeezing operator $\hat{S}(z)$ is defined for $z \in \mathbb{C}$ as

$$\hat{S}(z) \equiv \exp\left(\frac{1}{2}\left[z^* \hat{a}^2 - z \hat{a}^{\dagger 2}\right]\right). \tag{3}$$

   The thermal state $\theta(\bar{n})$ of mean photon number $\bar{n} \geq 0$ is defined as

$$\theta(\bar{n}) \equiv \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1}\right)^n |n\rangle\langle n|. \tag{4}$$

   Find the Hamiltonian matrix for the state $\hat{S}(z)\theta(\bar{n})\hat{S}^\dagger(z)$ as a function of z and $\bar{n}$ for $\bar{n} > 0$.

(b) Find the covariance matrix of the state $\hat{S}(z)\theta(\bar{n})\hat{S}^\dagger(z)$ as a function of z and $\bar{n}$ for $\bar{n} > 0$.

(c) The two-mode squeezing operator $S_2(z)$ is defined for $z \in \mathbb{C}$ as

$$\hat{S}_2(z) \equiv \exp\left(\frac{1}{2}\left[z^* \hat{a}\hat{b} - z\hat{a}^\dagger \hat{b}^\dagger\right]\right). \tag{5}$$

Find the Hamiltonian matrix of the state $\hat{S}_2(z)(\theta(\bar{n}_1) \otimes \theta(\bar{n}_2))\hat{S}_2^\dagger(z)$ as a function of z, $\bar{n}_1$, and $\bar{n}_2$ for $\bar{n}_1, \bar{n}_2 > 0$.

(d) Find the covariance matrix of the state $\hat{S}_2(z)(\theta(\bar{n}_1) \otimes \theta(\bar{n}_2))\hat{S}_2^\dagger(z)$ as a function of z, $\bar{n}_1$, and $\bar{n}_2$ for $\bar{n}_1, \bar{n}_2 > 0$.

6. Similar to how displacement operators compose nicely ($\hat{D}_{r_1}\hat{D}_{r_2} = \hat{D}_{r_1+r_2} e^{-\frac{i}{2} r_1^T \Omega r_2}$), it turns out that exponentials of quadratic forms compose nicely as well.

Let H^1 and H^2 be complex symmetric matrices. That is, they have complex entries and satisfy $H = H^T$ for T the ordinary matrix transpose (not the conjugate transpose).

(a) Prove that

$$\left[-\frac{1}{2}\hat{r}^T H^1 \hat{r}, -\frac{1}{2}\hat{r}^T H^2 \hat{r}\right] = -\frac{1}{2}\hat{r}^T H^3 \hat{r}, \tag{6}$$

for

$$H^3 = -i\big(H^1 \Omega H^2 - H^2 \Omega H^1\big). \tag{7}$$

(b) Prove that

$$\left[-i\Omega H^1, -i\Omega H^2\right] = -i\Omega H^3, \tag{8}$$

for H^3 as given above.

(c) Explain how to use these commutation relations and the Baker–Campbell–Hausdorff formula (as given at BCH), to conclude that the complex symmetric matrix H^4 satisfying

$$\exp(-i\Omega H^1)\exp(-i\Omega H^2) = \exp(-i\Omega H^4) \tag{9}$$

also satisfies

$$\exp\left(-\frac{1}{2}\hat{r}^T H^1 \hat{r}\right)\exp\left(-\frac{1}{2}\hat{r}^T H^2 \hat{r}\right) = \exp\left(-\frac{1}{2}\hat{r}^T H^4 \hat{r}\right). \tag{10}$$

7. BONUS: Let H denote a symmetric $2n \times 2n \times 2n$ rank-three tensor, which leads to the Hamiltonian operator

$$\hat{H} = \sum_{j,k,l} H_{j,k,l}\hat{r}_j \hat{r}_k \hat{r}_l. \tag{11}$$

What is the transformation realized by $\hat{H}$ on the $2n$-dimensional vector $\hat{r}$ of quadrature operators? That is, calculate

$$\exp(i\hat{H}t)\hat{r}\exp(-i\hat{H}t) \tag{12}$$

What is the transformation realized by a Hamiltonian operator defined from a symmetric rank-k tensor?