PHYS 7895 Spring 2019
Gaussian Quantum Information
Homework 1

Due Tuesday, 5 February 2019, in class

Exercises 2–6 taken from Ref. [3]

1. Consider the squeezed vacuum state $|\xi(r)\rangle$, defined as

$$|\xi(r)\rangle := \frac{1}{\sqrt{\cosh r}} \sum_{m=0}^{\infty} c_m(r)|m\rangle, \tag{1}$$

for $\xi = re^{i\theta}$, where $r \in [0,\infty)$, $\theta \in [0, 2\pi]$, and

$$c_m(r) := \begin{cases} (-1)^m \frac{\sqrt{(2m)!}}{2^m m!} e^{im\theta} (\tanh r)^m & \text{if } m \text{ is even} \\ 0 & \text{else} \end{cases} \tag{2}$$

(a) Prove that the limiting object $\lim_{r\to\infty} |\xi(r)\rangle$ is not in the Hilbert space (Hint: Prove that the coefficients $c_m(\infty)$ are not square summable).

(b) Prove that the sequence $\{|\xi(r)\rangle\}_{r\geq 0}$ is not a Cauchy sequence.

2. Prove the polarization identity. For $T \in \mathcal{L}(\mathcal{H})$ and $\phi, \psi \in \mathcal{H}$, we have that

$$\langle \phi | T\psi \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \psi + i^k \phi | T(\psi + i^k \phi) \rangle. \tag{3}$$

For $T = I$, this becomes

$$\langle \phi|\psi\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\|\psi + i^k\phi\right\|^2, \tag{4}$$

and allows you to determine all inner products if you know all of the norms.

3. Let λ be an eigenvalue of $T \in \mathcal{L}(\mathcal{H})$. Show that $|\lambda| \leq \|T\|$.

4. Unlike the set of bounded operators $\mathcal{L}(\mathcal{H})$, the set of self-adjoint operators $\mathcal{L}_s(\mathcal{H})$ is not an algebra. Prove the following: the product ST of two self-adjoint operators S, T is self-adjoint if and only if S and T commute.

5. Let T be a positive semi-definite operator and λ one of its eigenvalues. Prove that $\lambda \geq 0$.

6. Prove that $-\|T\|I \leq T \leq \|T\|I$ for all $T \in \mathcal{L}_s(\mathcal{H})$.