PHYS 7895: Quantum Information Theory Fall 2015

Lecture 27 — November 25, 2015

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1 Overview

In the last lecture, we showed how the protocol for entanglement-assisted classical communication generates a whole family of protocols in quantum Shannon theory.

In this lecture, we discuss communication trade-offs in quantum Shannon theory.

2 Trade-off Coding

Suppose that you are a communication engineer working at a quantum communication company named *EA-USA*. Suppose further that your company has made quite a profit from entanglement-assisted classical communication, beating out the communication rates that other companies can achieve simply because your company has been able to generate high-quality noiseless entanglement between several nodes in its network, while the competitors have not been able to do so. But now suppose that your customer base has become so large that there is not enough entanglement to support protocols that achieve the rates given in the entanglement-assisted classical capacity theorem. Your boss would like you to make the best of this situation, by determining the optimal rates of classical communication for a fixed entanglement budget. He is hoping that you will be able to design a protocol such that there will only be a slight decrease in communication rates. You tell him that you will do your best.

What should you do in this situation? Your first thought might be that we have already determined unassisted classical codes with a communication rate equal to the channel Holevo information $\chi(\mathcal{N})$ and we have also determined entanglement-assisted codes with a communication rate equal to the channel mutual information $I(\mathcal{N})$. It might seem that a reasonable strategy is to mix these two strategies, using some fraction λ of the channel uses for the unassisted classical code and the other fraction $1-\lambda$ of the channel uses for the entanglement-assisted code. This strategy achieves a rate of

$$\lambda\chi(\mathcal{N}) + (1-\lambda)\, I(\mathcal{N}), \tag{1}$$

and it has an error no larger than the sum of the errors of the individual codes (thus, this error vanishes asymptotically). Meanwhile, it consumes entanglement at a lower rate of $(1-\lambda)\, E$ ebits per channel use, if E is the amount of entanglement that the original protocol for entanglement-assisted classical communication consumes. This simple mixing strategy is known as *time sharing*. You figure this strategy might perform well, and you suggest it to your boss. After your boss reviews your proposal, he sends it back to you, telling you that he already thought of this solution and suggests that you are going to have to be a bit more clever—otherwise, he suspects that the existing customer base will notice the drop in communication rates.

Another strategy for communication is known as *trade-off coding*. We explore this strategy in the forthcoming section. Trade-off coding beats time sharing for many channels of interest, but for other channels, it just reduces to time sharing. It is not clear *a priori* how to determine which channels benefit from trade-off coding, but it certainly depends on the channel for which Alice and Bob are coding. The book follows up on the development here by demonstrating that this trade-off coding strategy is provably optimal for certain channels, and for general channels, it is optimal in the sense of regularized formulas. Trade-off coding is our best known way to deal with the above situation with a fixed entanglement budget, and your boss should be pleased with these results. Furthermore, we can upgrade the protocol outlined below to one that achieves entanglement-assisted communication of both classical and quantum information.

2.1 Trading between Unassisted and Assisted Classical Communication

We first show that the resource inequality given in the following theorem is achievable, and we follow up with an interpretation of it in the context of trade-off coding. We name the protocol *CE trade-off coding* because it captures the trade-off between classical communication and entanglement consumption.

Theorem 1 (CE Trade-off Coding)**.** *The following resource inequality corresponds to an achievable protocol for entanglement-assisted classical communication over a noisy quantum channel:*

$$\langle \mathcal{N} \rangle + H(A|X)_\rho \, [qq] \geq I(AX;B)_\rho \, [c \to c] \, , \tag{2}$$

where ρ_{XAB} is a state of the following form:

$$\rho_{XAB} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A' \to B}(\varphi^x_{AA'}), \tag{3}$$

and the states $\varphi^x_{AA'}$ are pure.

Proof. The proof of the above trade-off coding theorem exploits the direct parts of both the HSW coding theorem and the entanglement-assisted classical capacity theorem. In particular, we exploit the fact that the HSW codewords arise from strongly typical sequences and that the entanglement-assisted quantum codewords from are tensor power states after tracing over Bob's shares of the entanglement. Suppose that Alice and Bob exploit an HSW code for the channel $\mathcal{N}_{A' \to B}$. Such a code consists of a codebook $\left\{ \rho^{x^n(m)} \right\}_m$ with $\approx 2^{nI(X;B)_\rho}$ quantum codewords. The Holevo information $I(X;B)_\rho$ is with respect to some classical–quantum state ρ_{XB} where

$$\rho_{XB} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A' \to B}(\rho^x_{A'}), \tag{4}$$

and each codeword $\rho^{x^n(m)}$ is a tensor-product state of the form

$$\rho_{x^n(m)} = \rho^{x_1(m)} \otimes \rho^{x_2(m)} \otimes \cdots \otimes \rho^{x_n(m)}. \tag{5}$$

Corresponding to the codebook is some decoding POVM $\{\Lambda^m_{B^n}\}$, which Bob can employ to decode each codeword transmitted through the channel with arbitrarily high probability for all $\varepsilon > 0$:

$$\forall m \quad \text{Tr} \left\{ \Lambda^m_{B^n} \mathcal{N}_{A'^n \to B^n}(\rho^{x^n(m)}_{A'^n}) \right\} \geq 1 - \varepsilon. \tag{6}$$

Recall from the direct part of the HSW theorem that we select each codeword from the set of strongly typical sequences for the distribution $p_X(x)$. This implies that each classical codeword $x^n(m)$ has approximately $np_X(a_1)$ occurrences of the symbol $a_1 \in \mathcal{X}$, $np_X(a_2)$ occurrences of the symbol $a_2 \in \mathcal{X}$, and so on, for all letters in the alphabet $\mathcal{X}$. Without loss of generality and for simplicity, we assume that each codeword $x^n(m)$ has exactly these numbers of occurrences of the symbols in the alphabet $\mathcal{X}$. Then for any strongly typical sequence x^n, there exists some permutation π that arranges it in lexicographical order according to the alphabet $\mathcal{X}$. That is, this permutation arranges the sequence x^n into $|\mathcal{X}|$ blocks, each of length $np_X(a_1), \ldots, np_X(a_{|\mathcal{X}|})$:

$$\pi(x^n) = \underbrace{a_1 \cdots a_1}_{np_X(a_1)} \underbrace{a_2 \cdots a_2}_{np_X(a_2)} \cdots \underbrace{a_{|\mathcal{X}|} \cdots a_{|\mathcal{X}|}}_{np_X(a_{|\mathcal{X}|})}. \tag{7}$$

The same holds true for the corresponding permutation operator π applied to a quantum state ρ^{x^n} generated from a strongly typical sequence x^n:

$$\pi(\rho^{x^n}) = \underbrace{\rho^{a_1} \otimes \cdots \otimes \rho^{a_1}}_{np_X(a_1)} \otimes \underbrace{\rho^{a_2} \otimes \cdots \otimes \rho^{a_2}}_{np_X(a_2)} \otimes \cdots \otimes \underbrace{\rho^{a_{|\mathcal{X}|}} \otimes \cdots \otimes \rho^{a_{|\mathcal{X}|}}}_{np_X(a_{|\mathcal{X}|})}. \tag{8}$$

Now, we assume that n is quite large, so large that each of $np_X(a_1), \ldots, np_X(a_{|\mathcal{X}|})$ are large enough for the law of large numbers to come into play for each block in the permuted sequence $\pi(x^n)$ and tensor-product state $\pi(\rho^{x^n})$. Let $\varphi^x_{AA'}$ be a purification of each $\rho^x_{A'}$ in the ensemble $\{p_X(x), \rho^x_{A'}\}$, where we assume that Alice has access to system A' and Bob has access to A. Then, for every HSW quantum codeword $\rho^{x^n(m)}_{A'^n}$, there is some purification $\varphi^{x^n(m)}_{A^n A'^n}$, where

$$\varphi^{x^n(m)}_{A^n A'^n} \equiv \varphi^{x_1(m)}_{A_1 A'_1} \otimes \varphi^{x_2(m)}_{A_2 A'_2} \otimes \cdots \otimes \varphi^{x_n(m)}_{A_n A'_n}, \tag{9}$$

Alice has access to the systems $A'^n \equiv A'_1 \cdots A'_n$, and Bob has access to $A^n \equiv A_1 \cdots A_n$. Applying the permutation π to any purified tensor-product state φ^{x^n} gives

$$\pi\left(\varphi^{x^n}\right) = \underbrace{\varphi^{a_1} \otimes \cdots \otimes \varphi^{a_1}}_{np_X(a_1)} \otimes \underbrace{\varphi^{a_2} \otimes \cdots \otimes \varphi^{a_2}}_{np_X(a_2)} \otimes \cdots \otimes \underbrace{\varphi^{a_{|\mathcal{X}|}} \otimes \cdots \otimes \varphi^{a_{|\mathcal{X}|}}}_{np_X(a_{|\mathcal{X}|})}, \tag{10}$$

where we have assumed that the permutation applies on both the purification systems A^n and the systems A'^n. We can now formulate a strategy for trade-off coding. Alice begins with a standard classical sequence $\hat{x}^n$ that is in lexicographical order, having exactly $np_X(a_i)$ occurrences of the symbol $a_i \in \mathcal{X}$ (of the form in (7)). According to this sequence, she arranges the states $\{\varphi^{a_i}_{AA'}\}$ to be in $|\mathcal{X}|$ blocks, each of length $np_X(a_i)$—the resulting state is of the same form as in (10). Since $np_X(a_i)$ is large enough for the law of large numbers to come into play, for each block, there exists an entanglement-assisted classical code with $\approx 2^{nI(A;B)_{\mathcal{N}(\varphi^{a_i})}}$ entanglement-assisted quantum codewords, where the quantum mutual information $I(A;B)_{\mathcal{N}(\varphi^{a_i})}$ is with respect to the state $\mathcal{N}_{A' \to B}(\varphi^{a_i}_{AA'})$. Let $n_i \equiv np_X(a_i)$. Then each of these $|\mathcal{X}|$ entanglement-assisted classical codes consumes $n_i H(A)_{\varphi^{a_i}_A}$ ebits. The entanglement-assisted quantum codewords for each block are of the form

$$U_{A^{n_i}}(s(l_i))(\varphi^{a_i}_{A^{n_i} A'^{n_i}})U^{\dagger}_{A^{n_i}}(s(l_i)), \tag{11}$$

where l_i is a message in the message set of size $\approx 2^{nI(A;B)_{\varphi^{a_i}}}$, the state $\varphi^{a_i}_{A^{n_i} A'^{n_i}} = \varphi^{a_i}_{A_1 A'_1} \otimes \cdots \otimes \varphi^{a_i}_{A_{n_i} A'_{n_i}}$, and the unitaries $U_{A^{n_i}}(s(l_i))$ are of the form from the entanglement-assisted classical capacity theorem. Observe that the codewords in (11) are all equal to $\rho^{a_i}_{A'^{n_i}}$ after tracing over

Bob's systems A^{n_i}, regardless of the particular unitary that Alice applies. Alice then determines the permutation π_m needed to permute the standard sequence $\hat{x}^n$ to a codeword sequence $x^n(m)$, and she applies the permutation operator π_m to her systems A'^n so that her channel input density operator is the HSW quantum codeword $\rho_{A'^n}^{x^n(m)}$ (we are tracing over Bob's systems A^n and applying the aforementioned observation to obtain this result). She transmits her systems A'^n over the channel to Bob. If Bob ignores his share of the entanglement in A^n, the state that he receives from the channel is $\mathcal{N}_{A'^n \to B^n}(\rho_{A'^n}^{x^n(m)})$. He then applies his HSW measurement $\{\Lambda_{B^n}^m\}$ to the systems B^n received from the channel, and he determines the sequence $x^n(m)$, and hence the message m, with nearly unit probability. Also, this measurement has negligible disturbance on the state, so that the post-measurement state is $2\sqrt{\varepsilon}$-close in trace distance to the state that Alice transmitted through the channel (in what follows, we assume that the measurement does not change the state, and we collect error terms at the end of the proof). Now that he knows m, he applies the inverse permutation operator π_m^{-1} to his systems B^n, and we are assuming that he already has his share A^n of the entanglement arranged in lexicographical order according to the standard sequence $\hat{x}^n$. His state is then as follows:

$$\bigotimes_{i=1}^{|\mathcal{X}|} U_{A^{n_i}}(s(l_i)) \left(\varphi_{A^{n_1} A'^{n_1}}^{a_1}\right) U_{A^{n_i}}^{\dagger}(s(l_i)). \tag{12}$$

At this point, he can decode the message l_i in the ith block by performing a collective measurement on the systems $A^{n_i} A'^{n_i}$. He does this for each of the $|\mathcal{X}|$ entanglement-assisted classical codes, and this completes the protocol for trade-off coding. The total error accumulated in this protocol is no larger than the sum of ε for the first measurement, $2\sqrt{\varepsilon}$ for the disturbance of the state, and $|\mathcal{X}|\,\varepsilon$ for the error from the final measurement of the $|\mathcal{X}|$ blocks. The proof here assumes that every classical codeword $x^n(m)$ has exactly $np_X(a_i)$ occurrences of symbol $a_i \in \mathcal{X}$, but it is straightforward to modify the above protocol to allow for imprecision, i.e., if the codewords are δ-strongly typical. Figure 1 depicts this protocol for an example. We now show how the total rate of classical communication adds up to $I(AX;B)_\rho$ where ρ_{XAB} is a state of the form in (3). First, we can apply the chain rule for quantum mutual information to observe that the total rate $I(AX;B)_\rho$ is the sum of a Holevo information $I(X;B)_\rho$ and a classically conditioned quantum mutual information $I(A;B|X)_\rho$:

$$I(AX;B)_\rho = I(X;B)_\rho + I(A;B|X)_\rho. \tag{13}$$

They achieve the rate $I(X;B)_\rho$ because Bob first reliably decodes the HSW quantum codeword, of which there can be $\approx 2^{nI(X;B)}$. His next step is to permute and decode the $|\mathcal{X}|$ blocks, each consisting of an entanglement-assisted classical code on $\approx np_X(x)$ channel uses. Each entanglement-assisted classical code can communicate $np_X(x)I(A;B)_{\rho^x}$ bits while consuming $np_X(x)H(A)$ ebits. Thus, the total rate of classical communication for this last part is

$$\frac{\#\text{ of bits generated}}{\#\text{ of channel uses}} \approx \frac{\sum_x n\; p_X(x) I(A;B)_{\rho^x}}{\sum_x n\; p_X(x)} \tag{14}$$

$$= \sum_x p_X(x) I(A;B)_{\rho^x} \tag{15}$$

$$= I(A;B|X)_\rho, \tag{16}$$

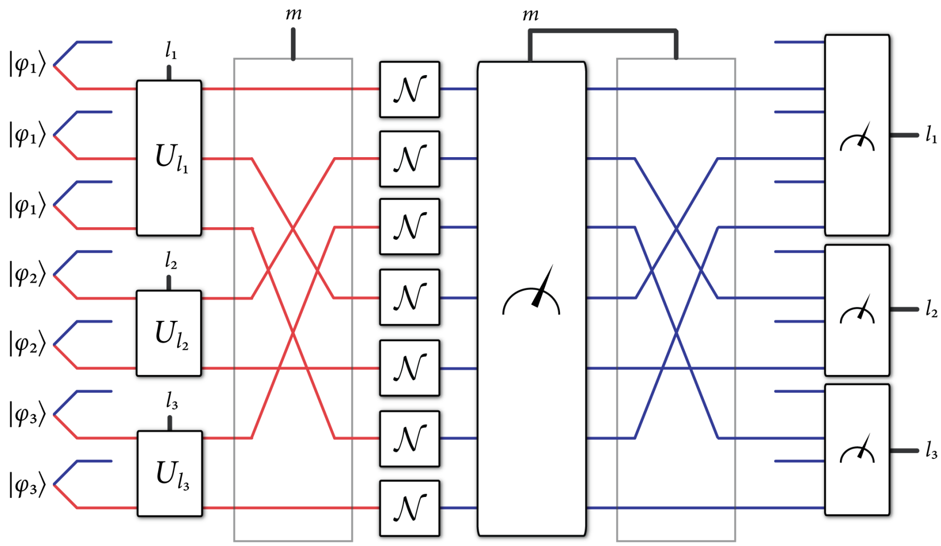

Figure 1: A simple protocol for trade-off coding between assisted and unassisted classical communication. Alice wishes to send the classical message m while also sending the messages l_1, l_2, and l_3. Her HSW codebook has the message m map to the sequence 1231213, which in turn gives the HSW quantum codeword $\rho^1 \otimes \rho^2 \otimes \rho^3 \otimes \rho^1 \otimes \rho^2 \otimes \rho^1 \otimes \rho^3$. A purification of these states is the following tensor product of pure states: $\varphi^1 \otimes \varphi^2 \otimes \varphi^3 \otimes \varphi^1 \otimes \varphi^2 \otimes \varphi^1 \otimes \varphi^3$, where Bob possesses the purification of each state in the tensor product. She begins with these states arranged in lexicographic order in three blocks (there are three letters in this alphabet). For each block i, she encodes the message l_i with the local unitaries for an entanglement-assisted classical code. She then permutes her shares of the entangled states according to the permutation associated with the message m. She inputs her systems to many uses of the channel, and Bob receives the outputs. His first action is to ignore his shares of the entanglement and perform a collective HSW measurement on all of the channel outputs. With high probability, he can determine the message m while causing a negligible disturbance to the state of the channel outputs. Based on the message m, he performs the inverse of the permutation that Alice used at the encoder. He combines his shares of the entanglement with the permuted channel outputs. His final three measurements are those given by the three entanglement-assisted codes Alice used at the encoder, and they detect the messages l_1, l_2, and l_3 with high probability.

and similarly, the total rate of entanglement consumption is

$$\frac{\#\text{ of ebit consumed}}{\#\text{ of channel uses}} \approx \frac{\sum_x n\, p_X(x) H(A)_{\rho^x}}{\sum_x n\, p_X(x)} \tag{17}$$

$$= \sum_x p_X(x) H(A)_{\rho^x} \tag{18}$$

$$= H(A|X)_\rho . \tag{19}$$

This gives the resource inequality in the statement of the theorem. □

2.2 Trading between Coherent and Classical Communication

We obtain the following corollary of Theorem 1, simply by upgrading the $|\mathcal{X}|$ entanglement-assisted classical codes to entanglement-assisted coherent codes. The upgrading is along the same lines as that which we did before, and for this reason, we omit the proof.

Corollary 2. *The following resource inequality corresponds to an achievable protocol for entanglement-assisted coherent communication over a noisy quantum channel* $\mathcal{N}$*:*

$$\langle \mathcal{N} \rangle + H(A|X)_\rho \, [qq] \geq I(A;B|X)_\rho \, [q \to qq] + I(X;B)_\rho \, [c \to c] \, , \tag{20}$$

where ρ_{XAB} *is a state of the following form:*

$$\rho_{XAB} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A' \to B}(\varphi^x_{AA'}), \tag{21}$$

and the states $\varphi^x_{AA'}$ *are pure.*

2.3 Trading between Classical Communication and Entanglement-Assisted Quantum Communication

We end this section with a protocol that achieves entanglement-assisted communication of both classical and quantum information. It is essential to the trade-off between a noisy quantum channel and the three resources of noiseless classical communication, noiseless quantum communication, and noiseless entanglement. We study this trade-off in full detail in the book, where we show that combining this protocol with teleportation, super-dense coding, and entanglement distribution is sufficient to achieve any task in dynamic quantum Shannon theory involving the three unit resources.

Corollary 3 (CQE Trade-off Coding)**.** *The following resource inequality corresponds to an achievable protocol for entanglement-assisted communication of classical and quantum information over a noisy quantum channel:*

$$\langle \mathcal{N} \rangle + \frac{1}{2} I(A;E|X)_\rho \, [qq] \geq \frac{1}{2} I(A;B|X)_\rho \, [q \to q] + I(X;B)_\rho \, [c \to c] \, , \tag{22}$$

where ρ_{XAB} *is a state of the following form:*

$$\rho_{XABE} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes U^{\mathcal{N}}_{A' \to BE}(\varphi^x_{AA'}), \tag{23}$$

the states $\varphi^x_{AA'}$ *are pure, and* $U^{\mathcal{N}}_{A' \to BE}$ *is an isometric extension of the channel* $\mathcal{N}_{A' \to B}$*.*

Proof. Consider the following chain of resource inequalities:

$$\begin{aligned}
&\langle \mathcal{N} \rangle + H(A|X)_\rho\,[qq] \\
&\geq I(A;B|X)_\rho\,[q \to qq] + I(X;B)_\rho\,[c \to c] && (24)\\
&\geq \frac{1}{2} I(A;B|X)_\rho\,[qq] + \frac{1}{2} I(A;B|X)_\rho\,[q \to q] + I(X;B)_\rho\,[c \to c]\,. && (25)
\end{aligned}$$

The first inequality is the statement in Corollary 2, and the second inequality follows from the coherent communication identity. After resource cancelation and noting that $H(A|X)_\rho - \frac{1}{2} I(A;B|X)_\rho = \frac{1}{2} I(A;E|X)_\rho$, the resulting resource inequality is equivalent to the one in (22). □

2.4 Trading between Classical and Quantum Communication

Our final trade-off coding protocol that we consider is that between classical and quantum communication. The proof of the below resource inequality follows by combining the protocol in Corollary 3 with entanglement distribution. Thus, we omit the proof.

Corollary 4 (CQ Trade-off Coding)**.** *The following resource inequality corresponds to an achievable protocol for simultaneous classical and quantum communication over a noisy quantum channel:*

$$\langle \mathcal{N} \rangle \geq I(A\rangle BX)_\rho\,[q \to q] + I(X;B)_\rho\,[c \to c]\,, \tag{26}$$

where ρ_{XAB} is a state of the following form:

$$\rho_{XAB} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A' \to B}(\varphi^x_{AA'}), \tag{27}$$

and the states $\varphi^x_{AA'}$ are pure.

This chapter unifies all of the channel coding theorems that we have studied in this book. One of the most general information-processing tasks that a sender and receiver can accomplish is to transmit classical and quantum information and generate entanglement with many independent uses of a quantum channel and with the assistance of classical communication, quantum communication, and shared entanglement.[1] The resulting rates for communication are *net* rates that give the generation rate of a resource less its consumption rate. Since we have three resources, all achievable rates are rate triples (C, Q, E) that lie in a three-dimensional capacity region, where C is the net rate of classical communication, Q is the net rate of quantum communication, and E is the net rate of entanglement consumption/generation. The capacity theorem for this general scenario is known as the quantum dynamic capacity theorem, and it is the main theorem that we prove in this chapter. All of the rates given in the channel coding theorems of previous chapters are special points in this three-dimensional capacity region.

The proof of the quantum dynamic capacity theorem comes in two parts: the direct coding theorem and the converse theorem. The direct coding theorem demonstrates that the strategy for achieving any point in the three-dimensional capacity region is remarkably simple: we just combine the protocol from Corollary 3 for entanglement-assisted classical and quantum communication with the three unit protocols of teleportation, super-dense coding, and entanglement distribution. The

[1] Recall that the book addressed a special case of this information processing task that applies to the scenario in which the sender and receiver do not have access to many independent uses of a noisy quantum channel.

interpretation of the achievable rate region is that it is the unit resource capacity region from the book translated along the points achievable with the protocol from Corollary 3. The proof of the converse theorem is perhaps the more difficult part—we analyze the most general protocol that can consume and generate classical communication, quantum communication, and entanglement along with the consumption of many independent uses of a quantum channel, and we show that the net rates for such a protocol are bounded by the achievable rate region. In the general case, our characterization is multi-letter, meaning that the computation of the capacity region requires an optimization over a potentially infinite number of channel uses and is thus intractable. However, the quantum Hadamard channels are a special class of channels for which the regularization is not necessary, and we can compute their capacity regions over a single instance of the channel. Another important class of channels for which the capacity region is known is the class of lossy bosonic channels (though the optimality proof is only up to a long-standing conjecture which many researchers believe to be true). These lossy bosonic channels model free-space communication or loss in a fiber optic cable and thus have an elevated impetus for study because of their importance in practical applications.

One of the most important questions for communication in this three-dimensional setting is whether it is really necessary to exploit the trade-off coding strategy given in Corollary 3. That is, would it be best simply to use a classical communication code for a fraction of the channel uses, a quantum communication code for another fraction, an entanglement-assisted code for another fraction, etc.? Such a strategy is known as time sharing and allows the sender and receiver to achieve convex combinations of any rate triples in the capacity region. The answer to this question depends on the channel. For example, time sharing is optimal for the quantum erasure channel, but it is not for a dephasing channel or a lossy bosonic channel. In fact, trade-off coding for a lossy bosonic channel can give tremendous performance gains over time sharing. How can we know which one will perform better in the general case? It is hard to say, but at the very least, we know that time sharing is a special case of trade-off coding as we argued in the book. Thus, from this perspective, it might make sense simply to always use a trade-off strategy.

We organize this chapter as follows. We first review the information-processing task corresponding to the quantum dynamic capacity region. Section 4 states the quantum dynamic capacity theorem and shows how many of the capacity theorems we studied previously arise as special cases of it. The next two sections prove the direct coding theorem and the converse theorem. The book introduces the quantum dynamic capacity formula, which is important for analyzing whether the quantum dynamic capacity region is single-letter. In the final section of this chapter, we compute and plot the quantum dynamic capacity region for the dephasing channels and the lossy bosonic channels.

3 The Information-Processing Task

Figure 2 depicts the most general protocol for generating classical communication, quantum communication, and entanglement with the consumption of a noisy quantum channel $\mathcal{N}_{A'\to B}$ and the same respective resources. Alice possesses two classical registers (each labeled by M and of dimension $2^{n\bar{C}}$), a quantum register A_1 of dimension $2^{n\bar{Q}}$ entangled with a reference system R, and another quantum register T_A of dimension $2^{n\tilde{E}}$ that contains her share of the shared entanglement with Bob:

$$\omega_{MMRA_1T_AT_B} \equiv \overline{\Phi}_{MM} \otimes \Phi_{RA_1} \otimes \Phi_{T_AT_B}. \tag{28}$$

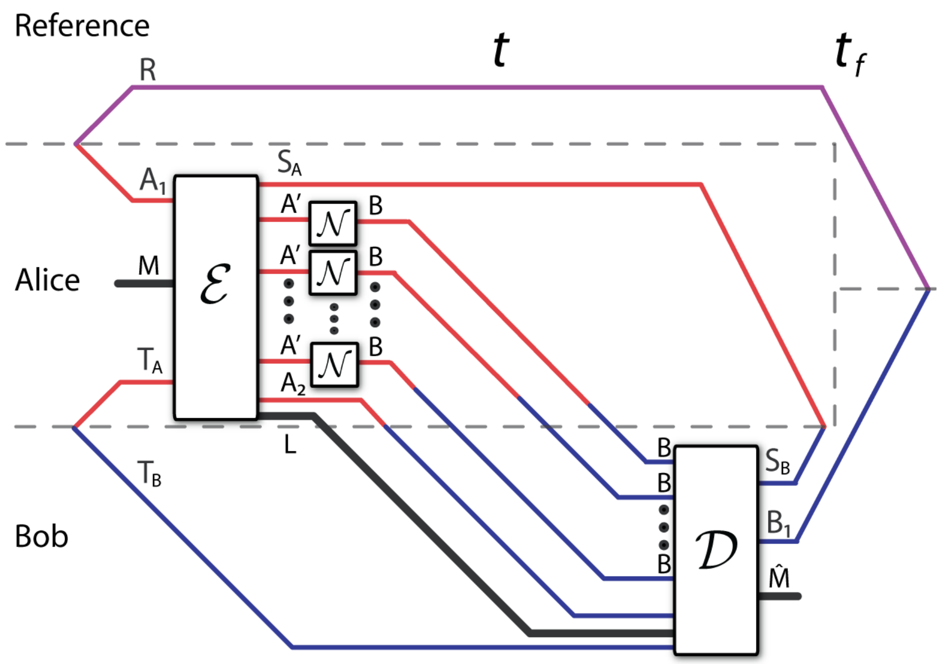

Figure 2: The most general protocol for generating classical communication, quantum communication, and entanglement generation with the help of the same respective resources and many uses of a noisy quantum channel. Alice begins with her classical register M, her quantum register A_1, and her half of the shared entanglement in register T_A. She encodes according to some CPTP map $\mathcal{E}$ that outputs a quantum register S_A, many registers A'^n, a quantum register A_2, and a classical register L. She inputs A'^n to many uses of the noisy channel $\mathcal{N}$ and transmits A_2 over a noiseless quantum channel and L over a noiseless classical channel. Bob receives the channel outputs B^n, the quantum register A_2, and the classical register L and performs a decoding $\mathcal{D}$ that recovers the quantum information and classical message. The decoding also generates entanglement with system S_A. Many protocols are a special case of the above one. For example, the protocol is entanglement-assisted communication of classical and quantum information if the registers L, S_A, S_B, and A_2 are null.

She passes one of the classical registers and the registers A_1 and T_A into a CPTP encoding map $\mathcal{E}_{MA_1T_A\to A'^n S_A L A_2}$ that outputs a quantum register S_A of dimension $2^{n\bar{E}}$ and a quantum register A_2 of dimension $2^{n\tilde{Q}}$, a classical register L of dimension $2^{n\tilde{C}}$, and many quantum systems A'^n for input to the channel. The register S_A is for creating entanglement with Bob. The state after the encoding map $\mathcal{E}$ is as follows:

$$\omega_{MA'^n S_A L A_2 R T_B} \equiv \mathcal{E}_{MA_1T_A\to A'^n S_A L A_2}(\omega_{MMRA_1T_AT_B}). \tag{29}$$

She sends the systems A'^n through many uses $\mathcal{N}_{A'^n\to B^n}$ of the noisy channel $\mathcal{N}_{A'\to B}$, transmits L over a noiseless classical channel, and transmits A_2 over a noiseless quantum channel, producing the following state:

$$\omega_{MB^n S_A L A_2 R T_B} \equiv \mathcal{N}_{A'^n\to B^n}(\omega_{MA'^n S_A L A_2 R T_B}). \tag{30}$$

The above state is a state of the following form:

$$\sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}_{A'^n\to B^n}(\rho^x_{AA'^n}), \tag{31}$$

with $A \equiv RT_BA_2S_A$ and $X \equiv ML$. Bob then applies a map $\mathcal{D}_{B^nA_2T_BL\to B_1S_B\hat{M}}$ that outputs a quantum system B_1, a quantum system S_B, and a classical register $\hat{M}$. Let ω' denote the final state. The following condition holds for a good protocol:

$$\left\|\overline{\Phi}_{M\hat{M}} \otimes \Phi_{RB_1} \otimes \Phi_{S_AS_B} - \omega'_{MB_1S_B\hat{M}S_AR}\right\|_1 \leq \varepsilon, \tag{32}$$

implying that Alice and Bob establish maximal classical correlations in M and $\hat{M}$ and maximal entanglement between S_A and S_B. The above condition also implies that the coding scheme preserves the entanglement with the reference system R. The net rate triple for the protocol is as follows: $\left(\bar{C}-\tilde{C}-\delta, \bar{Q}-\tilde{Q}-\delta, \bar{E}-\tilde{E}-\delta\right)$ for some arbitrarily small $\delta > 0$. The protocol generates a resource if its corresponding rate is positive, and it consumes a resource if its corresponding rate is negative. We say that a rate triple (C, Q, E) is achievable if there exists a protocol of the above form for all $\delta, \varepsilon > 0$ and sufficiently large n.

4 The Quantum Dynamic Capacity Theorem

The dynamic capacity theorem gives bounds on the reliable communication rates of a noisy quantum channel when combined with the noiseless resources of classical communication, quantum communication, and shared entanglement. The theorem applies regardless of whether a protocol consumes the noiseless resources or generates them.

Theorem 5 (Quantum Dynamic Capacity). *The dynamic capacity region $\mathcal{C}_{\text{CQE}}(\mathcal{N})$ of a quantum channel $\mathcal{N}$ is equal to the following expression:*

$$\mathcal{C}_{\text{CQE}}(\mathcal{N}) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k}\mathcal{C}^{(1)}_{\text{CQE}}(\mathcal{N}^{\otimes k})}, \tag{33}$$

where the overbar indicates the closure of a set. The region $\mathcal{C}^{(1)}_{\text{CQE}}(\mathcal{N})$ is the union of the state-dependent regions $\mathcal{C}^{(1)}_{\text{CQE},\sigma}(\mathcal{N})$:

$$\mathcal{C}^{(1)}_{\text{CQE}}(\mathcal{N}) \equiv \bigcup_{\sigma} \mathcal{C}^{(1)}_{\text{CQE},\sigma}(\mathcal{N}). \tag{34}$$

The state-dependent region $\mathcal{C}^{(1)}_{\mathrm{CQE},\sigma}(\mathcal{N})$ *is the set of all rates* C*,* Q*, and* E*, such that*

$$C + 2Q \leq I(AX;B)_\sigma, \tag{35}$$
$$Q + E \leq I(A\rangle BX)_\sigma, \tag{36}$$
$$C + Q + E \leq I(X;B)_\sigma + I(A\rangle BX)_\sigma. \tag{37}$$

The above entropic quantities are with respect to a classical–quantum state σ_{XAB} *where*

$$\sigma_{XAB} \equiv \sum_x p_X(x)|x\rangle\langle x|_X \otimes \mathcal{N}_{A'\to B}(\phi^x_{AA'}), \tag{38}$$

and the states $\phi^x_{AA'}$ *are pure. It is implicit that one should consider states on* A'^k *instead of* A' *when taking the regularization in* (33)*.*

The above theorem is a "multi-letter" capacity theorem because of the regularization in (33)). However, we show in the book that the regularization is not necessary for the Hadamard class of channels. We prove the above theorem in two parts:

1. The direct coding theorem in the book shows that combining the protocol from Corollary 3 with teleportation, super-dense coding, and entanglement distribution achieves the above region.

2. The converse theorem in Section **??** demonstrates that any coding scheme cannot do better than the regularization in (33), in the sense that a scheme with vanishing error should have its rates below the above amounts.

5 The Direct Coding Theorem

The unit resource achievable region is what Alice and Bob can achieve with the protocols entanglement distribution, teleportation, and super-dense coding. It is the cone of the rate triples corresponding to these protocols:

$$\{\alpha\,(0,-1,1) + \beta\,(2,-1,-1) + \gamma\,(-2,1,-1) : \alpha,\beta,\gamma \geq 0\}\,. \tag{39}$$

We can also write any rate triple (C,Q,E) in the unit resource capacity region with a matrix equation:

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}. \tag{40}$$

The inverse of the above matrix is as follows:

$$\begin{bmatrix} -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \tag{41}$$

and gives the following set of inequalities for the unit resource achievable region:

$$C + 2Q \leq 0, \tag{42}$$
$$Q + E \leq 0, \tag{43}$$
$$C + Q + E \leq 0, \tag{44}$$

by inverting the matrix equation in (40) and applying the constraints $\alpha, \beta, \gamma \geq 0$.

Now, let us include the protocol from Corollary 3 for entanglement-assisted communication of classical and quantum information. Corollary 3 states that we can achieve the following rate triple by channel coding over a noisy quantum channel $\mathcal{N}_{A' \to B}$:

$$\left(I(X;B)_\sigma, \frac{1}{2} I(A;B|X)_\sigma, -\frac{1}{2} I(A;E|X)_\sigma \right), \tag{45}$$

for any state σ_{XABE} of the form

$$\sigma_{XABE} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes U^{\mathcal{N}}_{A' \to BE}(\phi^x_{AA'}), \tag{46}$$

where $U^{\mathcal{N}}_{A' \to BE}$ is an isometric extension of the quantum channel $\mathcal{N}_{A' \to B}$. Specifically, we showed in Corollary 3 that one can achieve the above rates with vanishing error in the limit of large blocklength. Thus the achievable rate region is the following translation of the unit resource achievable region in (40):

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} I(X;B)_\sigma \\ \frac{1}{2} I(A;B|X)_\sigma \\ -\frac{1}{2} I(A;E|X)_\sigma \end{bmatrix}. \tag{47}$$

We can now determine bounds on an achievable rate region that employs the above coding strategy. We apply the inverse of the matrix in (40) to the left-hand side and right-hand side, giving

$$\begin{bmatrix} -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} C \\ Q \\ E \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} I(X;B)_\sigma \\ \frac{1}{2} I(A;B|X)_\sigma \\ -\frac{1}{2} I(A;E|X)_\sigma \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}. \tag{48}$$

Then using the following identities:

$$I(X;B)_\sigma + I(A;B|X)_\sigma = I(AX;B)_\sigma, \tag{49}$$

$$\frac{1}{2} I(A;B|X)_\sigma - \frac{1}{2} I(A;E|X)_\sigma = I(A\rangle BX)_\sigma, \tag{50}$$

and the constraints $\alpha, \beta, \gamma \geq 0$, we obtain the inequalities in (35)–(37), corresponding exactly to the state-dependent region in Theorem 5. Taking the union over all possible states σ in (38)) and taking the regularization gives the full dynamic achievable rate region.

Figure 3 illustrates an example of the general polyhedron specified by (35)–(37), where the channel is the qubit dephasing channel $\rho \to (1-p)\rho + pZ\rho Z$ with dephasing parameter $p = 0.2$, and the input state is

$$\sigma_{XAA'} \equiv \frac{1}{2} (|0\rangle\langle 0|_X \otimes \phi^0_{AA'} + |1\rangle\langle 1|_X \otimes \phi^1_{AA'}), \tag{51}$$

where

$$|\phi^0\rangle_{AA'} \equiv \sqrt{1/4} |00\rangle_{AA'} + \sqrt{3/4} |11\rangle_{AA'}, \tag{52}$$

$$|\phi^1\rangle_{AA'} \equiv \sqrt{3/4} |00\rangle_{AA'} + \sqrt{1/4} |11\rangle_{AA'}. \tag{53}$$

The state σ_{XABE} resulting from the channel is $U^{\mathcal{N}}_{A' \to BE}(\sigma_{XAA'})$ where $U^{\mathcal{N}}_{A' \to BE}$ is an isometric extension of the qubit dephasing channel. The figure caption provides a detailed explanation of the state-dependent region $\mathcal{C}^{(1)}_{\text{CQE},\sigma}$ (note that Figure 3 displays the state-dependent region and does not display the full capacity region).

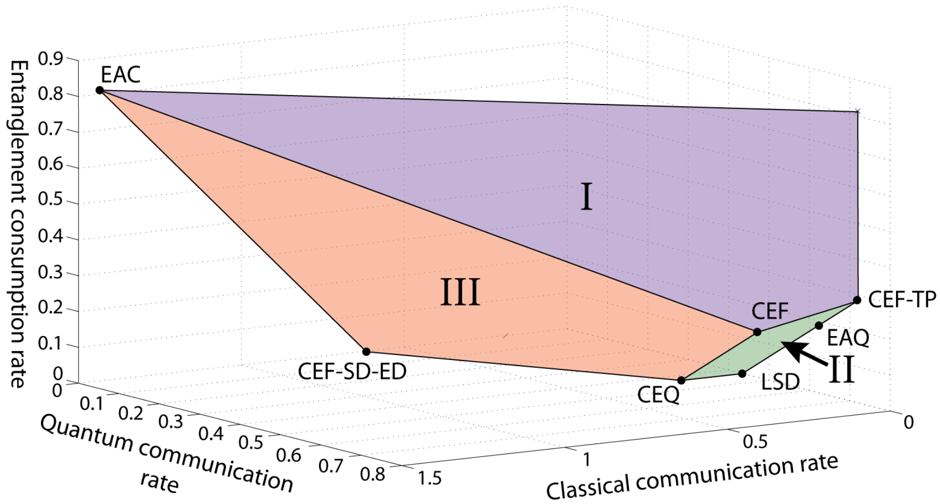

Figure 3: An example of the state-dependent achievable region $\mathcal{C}^{(1)}_{\text{CQE}\,\sigma}(\mathcal{N})$ corresponding to a state σ_{XABE} that arises from a qubit dephasing channel with dephasing parameter $p = 0.2$. The figure depicts the octant corresponding to the consumption of entanglement and the generation of classical and quantum communication. The state input to the channel $\mathcal{N}$ is $\sigma_{XAA'}$, defined in (51). The plot features seven achievable corner points of the state-dependent region. We can achieve the convex hull of these seven points by time sharing any two different coding strategies. We can also achieve any point above an achievable point by consuming more entanglement than necessary. The seven achievable points correspond to entanglement-assisted quantum communication (EAQ), the protocol from Corollary 4 for classically enhanced quantum communication (CEQ), the protocol from Theorem 1 for entanglement-assisted classical communication with limited entanglement (EAC), quantum communication (LSD), combining CEF with entanglement distribution and super-dense coding (CEF-SD-ED), the protocol from Corollary 3 for entanglement-assisted communication of classical and quantum information (CEF), and combining CEF with teleportation (CEF-TP). Observe that we can obtain EAC by combining CEF with super-dense coding, so that the points CEQ, CEF, EAC, and CEF-SD-ED all lie in plane III. Observe that we can obtain CEQ from CEF by entanglement distribution and we can obtain LSD from EAQ and EAQ from CEF-TP, both by entanglement distribution. Thus, the points CEF, CEQ, LSD, EAQ, and CEF-TP all lie in plane II. Finally, observe that we can obtain all corner points by combining CEF with the unit protocols of teleportation, super-dense coding, and entanglement distribution. The bounds in (35)–(37) uniquely specify the respective planes I-III. We obtain the full achievable region by taking the union over all states σ of the state-dependent regions $\mathcal{C}^{(1)}_{\sigma}(\mathcal{N})$ and taking the regularization, as outlined in Theorem 5. The above region is a translation of the unit resource capacity region from Chapter **??** to the protocol for entanglement-assisted communication of classical and quantum information.