PHYS 7895: Quantum Information Theory Fall 2015

Lecture 26 — November 23, 2015

Prof. Mark M. Wilde *Scribe: Mark M. Wilde*

1 Overview

In the last lecture, we proved the entanglement-assisted classical capacity theorem.

In this lecture, we discuss how to use this result to recover many protocols in quantum Shannon theory.

2 Introduction

This chapter demonstrates the power of both coherent communication and the particular protocol for entanglement-assisted classical coding from the previous chapter. Recall that coherent dense coding is a version of the dense coding protocol in which the sender and receiver perform all of its steps coherently.[1] Since our protocol for entanglement-assisted classical coding from the previous chapter is really just a glorified dense coding protocol, the sender and receiver can perform each of its steps coherently, generating a protocol for entanglement-assisted coherent coding. Then, by exploiting the fact that two coherent bits are equivalent to a qubit and an ebit, we obtain a protocol for entanglement-assisted quantum coding that consumes far less entanglement than a naive strategy would in order to accomplish this task. We next combine this entanglement-assisted quantum coding protocol with entanglement distribution and obtain a protocol for which the channel's coherent information is an achievable rate for quantum communication. This sequence of steps demonstrates an alternate proof of the direct part of the quantum channel coding theorem, which is given in the book.

Entanglement-assisted classical communication is one generalization of super-dense coding, in which the noiseless qubit channel becomes an arbitrary noisy quantum channel while the noiseless ebits remain noiseless. Another generalization of super-dense coding is a protocol named *noisy super-dense coding*, in which the shared entanglement becomes a shared noisy state ρ_{AB} and the noiseless qubit channels remain noiseless. Interestingly, the protocol that we employ in this chapter for noisy super-dense coding is essentially equivalent to the protocol from the previous chapter for entanglement-assisted classical communication, with some slight modifications to account for the different setting. We can also construct a coherent version of noisy super-dense coding, leading to a protocol that we name *coherent state transfer*. Coherent state transfer accomplishes not only the task of generating coherent communication between Alice and Bob, but it also allows Alice to transfer her share of the state ρ_{AB} to Bob. By combining coherent state transfer with

[1]Performing a protocol coherently means that we replace conditional unitaries with controlled unitaries and measurements with controlled gates.

both the coherent communication identity and teleportation, we obtain protocols for quantum-assisted state transfer and classical-assisted state transfer, respectively. The latter protocol gives an operational interpretation to the quantum conditional entropy $H(A|B)_\rho$—if it is positive, then the protocol consumes entanglement at the rate $H(A|B)_\rho$, and if it is negative, the protocol generates entanglement at the rate $|H(A|B)_\rho|$.

3 Entanglement-Assisted Quantum Communication

The entanglement-assisted classical capacity theorem states that the quantum mutual information of a channel is equal to its capacity for transmitting classical information with the help of shared entanglement, and the direct coding theorem provides a protocol that achieves the capacity. We were not much concerned with the rate at which this protocol consumes entanglement, but a direct calculation reveals that it consumes $H(A)_\varphi$ ebits per channel use, where $|\varphi\rangle_{AB}$ is the bipartite state that they share before the protocol begins.[2]

Suppose now that Alice is interested in exploiting the channel and shared entanglement in order to transmit quantum information to Bob. There is a simple (and as we will see, naive) way that we can convert the previous protocol to one that transmits quantum information: they can just combine it with teleportation. This naive strategy requires consuming ebits at an additional rate of $\frac{1}{2}I(A;B)_\rho$ in order to have enough entanglement to combine with teleportation, where $\rho_{AB} \equiv \mathcal{N}_{A'\to B}(\varphi_{AA'})$. To see this, consider the following resource inequalities:

$$\langle \mathcal{N} \rangle + \left(H(A)_\rho + \frac{1}{2} I(A;B)_\rho \right) [qq] \geq I(A;B)_\rho \, [c \to c] + \frac{1}{2} I(A;B)_\rho \, [qq] \tag{1}$$

$$\geq \frac{1}{2} I(A;B)_\rho \, [q \to q] \, . \tag{2}$$

The first inequality follows by having them exploit the channel and the $nH(A)_\rho$ ebits to generate classical communication at a rate $I(A;B)_\rho$ (while doing nothing with the extra $n\frac{1}{2}I(A;B)_\rho$ ebits). Alice then exploits the ebits and the classical communication in a teleportation protocol to send $n\frac{1}{2}I(A;B)_\rho$ qubits to Bob. This rate of quantum communication is provably optimal—were it not so, it would be possible to combine the protocol in (1)–(2) with super-dense coding and beat the optimal rate for classical communication given by the entanglement-assisted classical capacity theorem.

Although the above protocol achieves the entanglement-assisted quantum capacity, we are left thinking that the entanglement consumption rate of $H(A)_\rho + \frac{1}{2}I(A;B)_\rho$ ebits per channel use might be a bit more than necessary because teleportation and super-dense coding are not dual under resource reversal. That is, if we combine the protocol with super-dense coding and teleportation *ad infinitum*, then it consumes an infinite amount of entanglement. In practice, this "back and forth" with teleportation and super-dense coding would be a poor way to consume the precious resource of entanglement.

How might we make more judicious use of shared entanglement? Recall that coherent communication was helpful for doing so, at least in the noiseless case. A sender and receiver can combine

[2] This result follows because they can concentrate n copies of the state $|\varphi\rangle_{AB}$ to $nH(A)_\varphi$ ebits, as we learned before. Also, they can "dilute" $nH(A)_\varphi$ ebits to n copies of $|\varphi\rangle_{AB}$ with the help of a sublinear amount of classical communication that does not factor into the resource count (we have not studied the protocol for entanglement dilution).

coherent teleportation and coherent dense coding *ad infinitum* without any net loss in entanglement, essentially because these two protocols are dual under resource reversal. The following theorem shows how we can upgrade the previous protocol to one that generates coherent communication instead of just classical communication. The resulting protocol is one way to have a version of coherent dense coding in which one noiseless resource is replaced by a noisy one.

Theorem 1 (Entanglement-Assisted Coherent Communication)**.** *The following resource inequality corresponds to an achievable protocol for entanglement-assisted coherent communication over a noisy quantum channel:*

$$\langle \mathcal{N} \rangle + H(A)_\rho \, [qq] \geq I(A;B)_\rho \, [q \to qq] \, , \tag{3}$$

where $\rho_{AB} \equiv \mathcal{N}_{A' \to B}(\varphi_{AA'})$*.*

Proof. Suppose that Alice and Bob share many copies of some pure, bipartite entangled state $|\varphi\rangle_{AB}$. Consider the code from the direct coding theorem in the book. We can say that it is a set of $D^2 \approx 2^{nI(A;B)_\rho}$ unitaries $U(s(m))$, from which Alice can select, and she applies a particular unitary $U(s(m))$ to her share A^n of the entanglement in order to encode message m. Also, Bob has a detection POVM $\left\{ \Lambda^m_{B'^n B^n} \right\}$ acting on his share of the entanglement and the channel outputs that he can exploit to detect message m. Just as we were able to construct a coherent super-dense coding protocol by performing all the steps in dense coding coherently, we can do so for the entanglement-assisted classical coding protocol. We track the steps in such a protocol. Suppose Alice shares a state with a reference system R to which she does not have access:

$$|\psi\rangle_{RA_1} \equiv \sum_{l,m=1}^{D^2} \alpha_{l,m} \, |l\rangle_R \, |m\rangle_{A_1}, \tag{4}$$

where $\{|l\rangle\}$ and $\{|m\rangle\}$ are some orthonormal bases for R and A_1, respectively. We say that Alice and Bob have implemented a coherent channel if they execute the map $|m\rangle_{A_1} \to |m\rangle_{A_1} |m\rangle_{B_1}$, which transforms the above state to

$$\sum_{l,m=1}^{D^2} \alpha_{l,m} \, |l\rangle_R \, |m\rangle_{A_1} \, |m\rangle_{B_1}. \tag{5}$$

We say that they have implemented a coherent channel *approximately* if the state resulting from the protocol is ε-close in trace distance to the above state. If we can show that ε is an arbitrary positive number that approaches zero in the asymptotic limit, then the simulation of an approximate coherent channel asymptotically becomes an exact simulation. Alice's first step is to append her shares of the entangled state $|\varphi\rangle_{A^n B^n}$ to $|\psi\rangle_{RA_1}$ and apply the following controlled unitary from her system A_1 to her system A^n:

$$\sum_m |m\rangle\langle m|_{A_1} \otimes U_{A^n}(s(m)). \tag{6}$$

The resulting global state is as follows:

$$\sum_{l,m} \alpha_{l,m} \, |l\rangle_R \, |m\rangle_{A_1} U_{A^n}(s(m)) |\varphi\rangle_{A^n B^n}. \tag{7}$$

By the structure of the unitaries $U(s(m))$ (see the book), the above state is equivalent to the following one:

$$\sum_{l,m} \alpha_{l,m} \left|l\right\rangle_R \left|m\right\rangle_{A_1} U^T_{B^n}(s(m)) |\varphi\rangle_{A^n B^n}. \tag{8}$$

Interestingly, observe that Alice applying the controlled gate in (6) is the same as her applying the non-local controlled gate $\sum_m |m\rangle \langle m|_{A_1} \otimes U^T_{B^n}(s(m))$, due to the non-local (and perhaps spooky!) properties of the entangled state $|\varphi\rangle_{A^n B^n}$. Alice then sends her systems A^n through many uses of the noisy quantum channel $\mathcal{N}_{A\to B'}$, whose isometric extension is $U^{\mathcal{N}}_{A\to B'E}$. Let $|\varphi\rangle_{B'^n E^n B^n}$ denote the state resulting from an isometric extension $U^{\mathcal{N}}_{A\to B'E}$ of the channel acting on the state $|\varphi\rangle_{A^n B^n}$:

$$|\varphi\rangle_{B'^n E^n B^n} \equiv U^{\mathcal{N}}_{A^n\to B'^n E^n} |\varphi\rangle_{A^n B^n}. \tag{9}$$

After Alice transmits through the channel, the state becomes

$$\sum_{l,m} \alpha_{l,m} \left|l\right\rangle_R \left|m\right\rangle_{A_1} U^T_{B^n}(s(m)) |\varphi\rangle_{B'^n E^n B^n}, \tag{10}$$

where Bob now holds his shares B^n of the entanglement and the channel outputs B'^n. (Observe that the action of the controlled unitary in (6) commutes with the action of the channel.) Rather than perform an incoherent measurement with the POVM $\left\{\Lambda^m_{B'^n B^n}\right\}$, Bob applies a coherent gentle measurement, an isometry of the following form:

$$\sum_m \sqrt{\Lambda^m_{B'^n B^n}} \otimes |m\rangle_{B_1}. \tag{11}$$

Using the result from the homework, we can readily check that the resulting state is $2\sqrt{\varepsilon}$-close in trace distance to the following state:

$$\sum_{l,m} \alpha_{l,m} \left|l\right\rangle_R \left|m\right\rangle_{A_1} U^T_{B^n}(s(m)) |\varphi\rangle_{B'^n E^n B^n} |m\rangle_{B_1}. \tag{12}$$

Thus, for the rest of the protocol, we pretend that they are acting on the above state. Alice and Bob would like to coherently remove the coupling of their index m to the environment, so Bob performs the following controlled unitary:

$$\sum_m |m\rangle\langle m|_{B_1} \otimes U^*_{B^n}(s(m)), \tag{13}$$

and the final state is

$$\sum_{l,m=1}^{D^2} \alpha_{l,m} \left|l\right\rangle_R \left|m\right\rangle_{A_1} |\varphi\rangle_{B'^n E^n B^n} |m\rangle_{B_1} = \left(\sum_{l,m=1}^{D^2} \alpha_{l,m} \left|l\right\rangle_R \left|m\right\rangle_{A_1} |m\rangle_{B_1}\right) \otimes |\varphi\rangle_{B'^n E^n B^n}. \tag{14}$$

Thus, this protocol implements a D^2-dimensional coherent channel up to an arbitrarily small error, and we have shown that the resource inequality in the statement of the theorem holds. Figure 1 depicts the entanglement-assisted coherent coding protocol. □

It is now a straightforward task to convert the protocol from Theorem 1 into one for entanglement-assisted quantum communication, by exploiting the coherent communication identity.

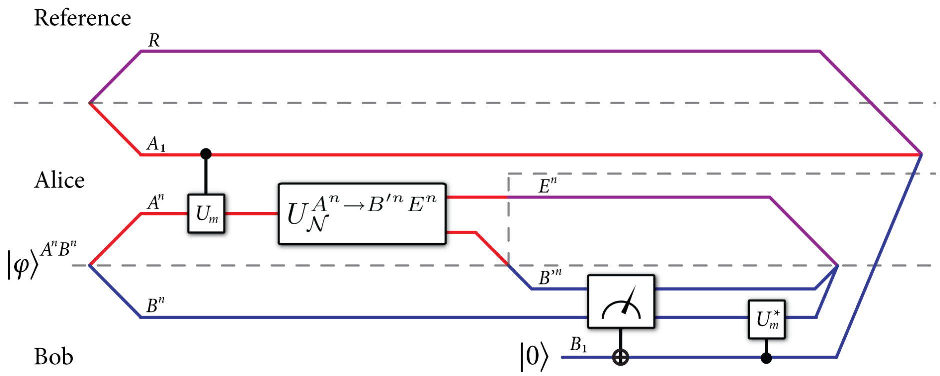

Figure 1: The protocol for entanglement-assisted coherent communication. Observe that it is the coherent version of the protocol for entanglement-assisted classical communication, just as coherent dense coding is the coherent version of super-dense coding. Instead of applying conditional unitaries, Alice applies a controlled unitary from her system A_1 to her share of the entanglement and sends the encoded state through many uses of the noisy channel. Rather than performing a POVM, Bob performs a coherent gentle measurement from his systems B'^n and B^n to an ancilla B_1. Finally, he applies a similar controlled unitary in order to decouple the environment from the state of his ancilla B_1.

Corollary 2 (Entanglement-Assisted Quantum Communication)**.** *The following resource inequality corresponds to an achievable protocol for entanglement-assisted quantum communication over a noisy quantum channel:*

$$\langle \mathcal{N} \rangle + \frac{1}{2} I(A;E)_\varphi \, [qq] \geq \frac{1}{2} I(A;B)_\varphi \, [q \to q] \, , \tag{15}$$

where $|\varphi\rangle_{ABE} \equiv U^{\mathcal{N}}_{A' \to BE} |\varphi\rangle_{AA'}$ *and* $U^{\mathcal{N}}_{A' \to BE}$ *is an isometric extension of the channel* $\mathcal{N}_{A' \to B}$*.*

Consider the coherent communication identity. This identity states that a D^2-dimensional coherent channel can perfectly simulate a D-dimensional quantum channel and a maximally entangled state $|\Phi\rangle_{AB}$ with Schmidt rank D. In terms of cobits, qubits, and ebits, the coherent communication identity is the following resource equality for D-dimensional systems:

$$2 \log D \, [q \to qq] = \log D \, [q \to q] + \log D \, [qq] \, . \tag{16}$$

Consider the following chain of resource inequalities:

$$\langle \mathcal{N} \rangle + H(A)_\varphi \, [qq] \geq I(A;B)_\varphi \, [q \to qq] \tag{17}$$

$$\geq \frac{1}{2} I(A;B)_\varphi \, [q \to q] + \frac{1}{2} I(A;B)_\varphi \, [qq] \, . \tag{18}$$

The first resource inequality is the statement of Theorem 1, and the second resource inequality follows from an application of coherent teleportation. If we then allow for catalytic protocols, in which we allow for some use of a resource with the demand that it be returned at the end of the protocol, we have a protocol for entanglement-assisted quantum communication:

$$\langle \mathcal{N} \rangle + \frac{1}{2} I(A;E)_\varphi \, [qq] \geq \frac{1}{2} I(A;B)_\varphi \, [q \to q] \, , \tag{19}$$

because $H(A)_\varphi - \frac{1}{2}I(A;B)_\varphi = \frac{1}{2}I(A;E)_\varphi$.

When comparing the entanglement consumption rate of the naive protocol in (1)–(2) with that of the protocol in Theorem 2, we see that the former requires an additional $I(A;B)_\rho$ ebits per channel use. Also, Theorem 2 leads to a simple proof of the achievability part of the quantum capacity theorem, as we see in the next section.

4 Quantum Communication

We can obtain a protocol for quantum communication simply by combining the protocol from Theorem 2 further with entanglement distribution. The resulting protocol again makes catalytic use of entanglement, in the sense that it exploits some amount of entanglement shared between Alice and Bob at the beginning of the protocol, but it generates the same amount of entanglement at the end, so that the net entanglement consumption rate of the protocol is zero.

Corollary 3 (Quantum Communication)**.** *The coherent information $Q(\mathcal{N})$ is an achievable rate for quantum communication over a quantum channel $\mathcal{N}$. That is, the following resource inequality holds:*

$$\langle \mathcal{N} \rangle \geq Q(\mathcal{N})\,[q \to q]\,, \tag{20}$$

where $Q(\mathcal{N}) \equiv \max_\varphi I(A\rangle B)_\rho$ and $\rho_{AB} \equiv \mathcal{N}_{A' \to B}(\varphi_{AA'})$.

Proof. If we further combine the entanglement-assisted quantum communication protocol from Theorem 2 with entanglement distribution at a rate $\frac{1}{2}I(A;E)_\rho$, we obtain the following resource inequalities:

$$\begin{aligned}
&\langle \mathcal{N} \rangle + \frac{1}{2}I(A;E)_\rho\,[qq] \\
&\geq \frac{1}{2}\left[I(A;B)_\rho - I(A;E)_\rho\right][q \to q] + \frac{1}{2}I(A;E)_\rho\,[q \to q] && (21)\\
&\geq \frac{1}{2}\left[I(A;B)_\rho - I(A;E)_\rho\right][q \to q] + \frac{1}{2}I(A;E)_\rho\,[qq]\,, && (22)
\end{aligned}$$

which after resource cancelation, becomes

$$\langle \mathcal{N} \rangle \geq I(A\rangle B)_\rho\,[q \to q]\,, \tag{23}$$

because $I(A\rangle B)_\rho = \frac{1}{2}\left[I(A;B)_\rho - I(A;E)_\rho\right]$. They can achieve the coherent information of the channel simply by generating codes from the state $\varphi_{AA'}$ that maximizes the channel's coherent information. □

5 Noisy Super-Dense Coding

Recall that the resource inequality for super-dense coding is

$$[q \to q] + [qq] \geq 2\,[c \to c]\,. \tag{24}$$

The entanglement-assisted classical communication protocol from the previous chapter is one way to generalize this protocol to a noisy setting, simply by replacing the noiseless qubit channels

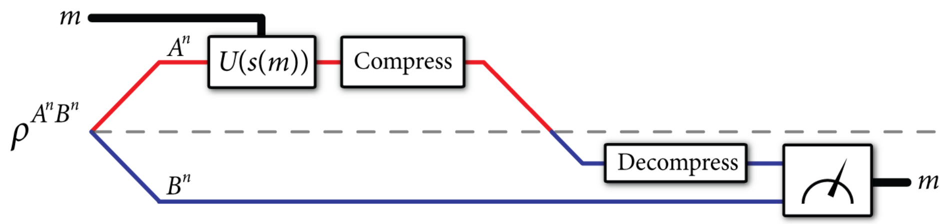

Figure 2: The protocol for noisy super-dense coding that corresponds to the resource inequality in Theorem 4. Alice first projects her share into its typical subspace (not depicted). She then applies a unitary encoding $U\left(s(m)\right)$, based on her message m, to her share of the state $\rho_{A^nB^n}$. She compresses her state to approximately $nH(A)_\rho$ qubits and transmits these qubits over noiseless qubit channels. Bob decompresses the state and performs a decoding POVM that gives Alice's message m with high probability.

in (24) with many uses of a noisy quantum channel. This replacement leads to the setting of entanglement-assisted classical communication presented in the previous chapter.

Another way to generalize super-dense coding is to let the entanglement be noisy while keeping the quantum channels noiseless. We allow Alice and Bob access to many copies of some shared noisy state ρ_{AB} and to many uses of a noiseless qubit channel with the goal of generating noiseless classical communication. One might expect the resulting protocol to be similar to that for entanglement-assisted classical communication, and this is indeed the case. The resulting protocol is known as *noisy super-dense coding*:

Theorem 4 (Noisy Super-Dense Coding)**.** *The following resource inequality corresponds to an achievable protocol for quantum-assisted classical communication with a noisy quantum state:*

$$\langle \rho_{AB} \rangle + H(A)_\rho \left[q \to q\right] \geq I(A;B)_\rho \left[c \to c\right], \tag{25}$$

where ρ_{AB} is some noisy bipartite state that Alice and Bob share at the beginning of the protocol.

Proof. Please see the book for a complete proof. The main idea is similar to that for the direct part of the entanglement-assisted capacity theorem. We just summarize the protocol. Alice and Bob begin with the state $\rho_{A^nB^n}$. Alice first performs a typical subspace measurement of her system A^n. This measurement succeeds with high probability and reduces the size of her system A^n to a subspace with size approximately equal to $nH(A)_\rho$ qubits. If Alice wishes to send message m, she applies the unitary $U_{A^n}(s(m))$ to her share of the state. She then performs a compression isometry from her subspace of A^n to $nH(A)_\rho$ qubits. She transmits her qubits over $nH(A)_\rho$ noiseless qubit channels, and Bob receives them. Bob performs the decompression isometry from the space of $nH(A)_\rho$ noiseless qubits to a space isomorphic to Alice's original systems A^n. He then performs the decoding POVM $\{\Lambda^m_{A^nB^n}\}$ and determines Alice's message m with vanishingly small error probability. Figure 2 depicts the protocol. □

6 State Transfer

We can also construct a coherent version of the noisy super-dense coding protocol, in a manner similar to the way in which the proof of Theorem 1 constructs a coherent version of entanglement-assisted classical communication. However, the coherent version of noisy super-dense coding achieves an additional task: the transfer of Alice's share of the state $(\rho_{AB})^{\otimes n}$ to Bob. The resulting protocol is known as coherent state transfer, and from this protocol, we can derive a protocol for quantum-communication-assisted state transfer, or quantum-assisted state transfer[3] for short.

Theorem 5 (Coherent State Transfer)**.** *The following resource inequality corresponds to an achievable protocol for coherent state transfer with a noisy state ρ_{AB}:*

$$\langle W_{S\to AB} : \rho_S\rangle + H(A)_\rho \, [q \to q] \geq I(A;B)_\rho \, [q \to qq] + \langle \mathrm{id}_{S\to \hat{B}B} : \rho_S\rangle, \tag{26}$$

where ρ_{AB} is some noisy bipartite state that Alice and Bob share at the beginning of the protocol.

The resource inequality in (42) features some notation that we have not seen yet. The expression $\langle W_{S\to AB} : \rho_S\rangle$ means that a source party S distributes many copies of the state ρ_S to Alice and Bob, by applying some isometry $W_{S\to AB}$ to the state ρ_S. This resource is effectively equivalent to Alice and Bob sharing many copies of the state ρ_{AB}, a resource we expressed in Theorem 4 as $\langle\rho_{AB}\rangle$. The expression $\langle \mathrm{id}_{S\to \hat{B}B} : \rho_S\rangle$ means that a source party applies the identity map to ρ_S and gives the full state to Bob. We can now state the meaning of the resource inequality in (42): Using n copies of the state ρ_{AB} and $nH(A)_\rho$ noiseless qubit channels, Alice can simulate $nI(A;B)_\rho$ noiseless coherent channels to Bob while at the same time transferring her share of the state $(\rho_{AB})^{\otimes n}$ to him.

Proof. A proof proceeds similarly to the proof of Theorem 1. Let $|\varphi\rangle_{ABR}$ be a purification of ρ_{AB}. Alice begins with a state that she shares with a reference system R_1, on which she would like to simulate coherent channels:

$$|\psi\rangle_{R_1 A_1} \equiv \sum_{l,m=1}^{D^2} \alpha_{l,m} \, |l\rangle_{R_1} \, |m\rangle_{A_1}, \tag{27}$$

where $D^2 \approx 2^{nI(A;B)_\rho}$. She appends $|\psi\rangle_{R_1A_1}$ to $|\varphi\rangle_{A^nB^nR^n} \equiv (|\varphi\rangle_{ABR})^{\otimes n}$ and applies a typical subspace measurement to her system A^n. (In what follows, we use the same notation for the typical projected state because the states are the same up to a vanishingly small error.) She applies the following controlled unitary to her systems A_1A^n:

$$\sum_m |m\rangle\langle m|_{A_1} \otimes U_{A^n}(s(m)), \tag{28}$$

resulting in the overall state

$$\sum_{l,m} \alpha_{l,m} \, |l\rangle_{R_1} \, |m\rangle_{A_1} \, U_{A^n}(s(m))|\varphi\rangle_{A^nB^nR^n}. \tag{29}$$

Alice compresses her A^n systems, sends them over $nH(A)_\rho$ noiseless qubit channels, and Bob receives them. He decompresses them and places them in systems $\hat{B}^n$ isomorphic to A^n. The resulting state is the same as $|\varphi\rangle_{A^nB^nR^n}$, with the systems A^n replaced by $\hat{B}^n$. Bob performs a coherent gentle measurement of the following form:

$$\sum_m \sqrt{\Lambda^m_{\hat{B}^nB^n}} \otimes |m\rangle_{B_1}, \tag{30}$$

[3]This protocol goes by several other names in the quantum Shannon theory literature: state transfer, fully quantum Slepian–Wolf, state merging, and the merging mother.

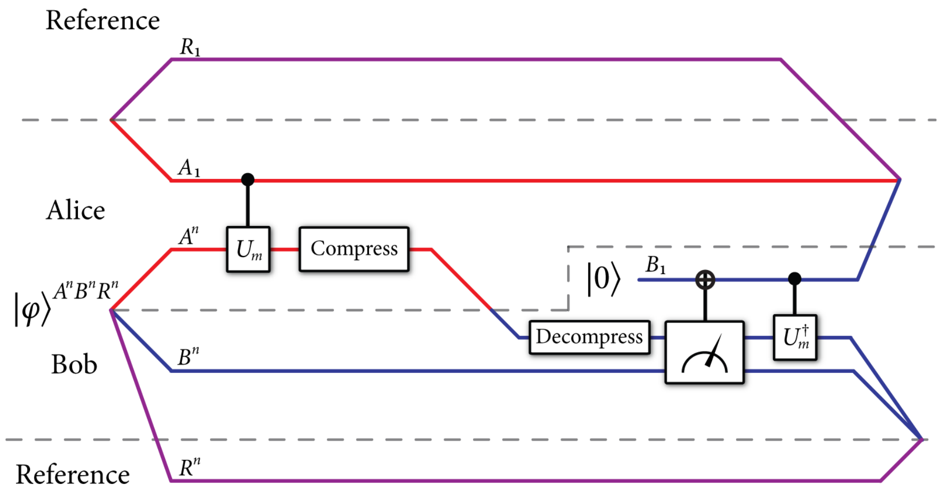

Figure 3: The protocol for coherent state transfer, a coherent version of the noisy super-dense coding protocol that accomplishes the task of state transfer in addition to coherent communication.

resulting in a state that is close in trace distance to

$$\sum_{l,m} \alpha_{l,m} \, |l\rangle_{R_1} \, |m\rangle_{A_1} \, |m\rangle_{B_1} U_{\hat{B}^n}(s(m)) |\varphi\rangle_{\hat{B}^n B^n R^n}. \tag{31}$$

He finally performs the controlled unitary

$$\sum_m |m\rangle\langle m|_{B_1} \otimes U^\dagger_{\hat{B}^n}(s(m)), \tag{32}$$

resulting in the state

$$\left(\sum_{l,m} \alpha_{l,m} \, |l\rangle_{R_1} \, |m\rangle_{A_1} \, |m\rangle_{B_1} \right) \otimes |\varphi\rangle_{\hat{B}^n B^n R^n}. \tag{33}$$

Thus, Alice has simulated $nI(A;B)_\rho$ coherent channels to Bob with arbitrarily small error, while also transferring her share of the state $|\varphi\rangle_{A^n B^n R^n}$ to him. Figure 3 depicts the protocol. □

We obtain the following resource inequality for quantum-assisted state transfer, by combining the above protocol with the coherent communication identity:

Corollary 6 (Quantum-Assisted State Transfer)**.** *The following resource inequality corresponds to an achievable protocol for quantum-assisted state transfer with a noisy state ρ_{AB}:*

$$\langle W_{S\to AB} : \rho_S \rangle + \frac{1}{2} I(A;R)_\varphi \, [q \to q] \geq \frac{1}{2} I(A;B)_\varphi \, [qq] + \langle \mathrm{id}_{S \to \hat{B}B} : \rho_S \rangle, \tag{34}$$

where ρ_{AB} is some noisy bipartite state that Alice and Bob share at the beginning of the protocol, and $|\varphi\rangle_{ABR}$ is a purification of it.

Proof. Consider the following chain of resource inequalities:

$$
\begin{aligned}
&\langle W_{S\to AB} : \rho_S\rangle + H(A)_\varphi\,[qq] \\
&\qquad \geq I(A;B)_\varphi\,[q\to qq] + \langle \mathrm{id}_{S\to \hat{B}B} : \rho_S\rangle && (35)\\
&\qquad \geq \frac{1}{2}I(A;B)_\varphi\,[q\to q] + \frac{1}{2}I(A;B)_\varphi\,[qq] + \langle \mathrm{id}_{S\to \hat{B}B} : \rho_S\rangle, && (36)
\end{aligned}
$$

where the first follows from coherent state transfer and the second follows from the coherent communication identity. By resource cancelation, we obtain the resource inequality in the statement of the theorem because $\frac{1}{2}I(A;R)_\varphi = H(A)_\rho - \frac{1}{2}I(A;B)_\rho$. □

Corollary 7 (Classical-Assisted State Transfer)**.** *The following resource inequality corresponds to an achievable protocol for classical-assisted state transfer with a noisy state ρ_{AB}:*

$$\langle W_{S\to AB} : \rho_S\rangle + I(A;R)_\varphi\,[c\to c] \geq I(A\rangle B)_\varphi\,[qq] + \langle \mathrm{id}_{S\to \hat{B}B} : \rho_S\rangle, \tag{37}$$

where ρ_{AB} is some noisy bipartite state that Alice and Bob share at the beginning of the protocol, and $|\varphi\rangle_{ABR}$ is a purification of it.

Proof. We simply combine the protocol above with teleportation:

$$
\begin{aligned}
&\langle W_{S\to AB} : \rho_S\rangle + \frac{1}{2}I(A;R)_\varphi\,[q\to q] + I(A;R)_\varphi\,[c\to c] + \frac{1}{2}I(A;R)_\varphi\,[qq] \\
&\qquad\qquad \geq \frac{1}{2}I(A;B)_\varphi\,[qq] + \langle \mathrm{id}_{S\to \hat{B}B} : \rho_S\rangle + \frac{1}{2}I(A;R)_\varphi\,[q\to q] && (38)
\end{aligned}
$$

Canceling terms for both quantum communication and entanglement, we obtain the resource inequality in the statement of the corollary. □

The above protocol gives a wonderful operational interpretation to the coherent information (or negative conditional entropy $-H(A|B)_\rho$). When the coherent information is positive, Alice and Bob share that rate of entanglement at the end of the protocol (and thus the ability to teleport if extra classical communication is available). When the coherent information is negative, they need to consume entanglement at a rate of $H(A|B)_\rho$ ebits per copy in order for the state transfer process to complete.