PHYS 7895: Quantum Information Theory Fall 2015

Lecture 13 — October 7, 2015

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1 Overview

In the last lecture, we discussed coherent communication and then moved on to the notion of a purification of a quantum state.

In this lecture, we continue with the theme of purification and develop the notion of an isometric extension of a quantum channel and related ideas.

2 Isometric Evolution

A quantum channel admits a purification as well. We motivate this idea with a simple example.

2.1 Example: Isometric Extension of the Bit-Flip Channel

Consider the bit-flip channel

$$\rho \to (1-p)\rho + pX\rho X. \tag{1}$$

It applies the identity operator with some probability $1-p$ and applies the bit-flip Pauli operator X with probability p. Suppose that we input a qubit system A in the state $|\psi\rangle$ to this channel. The ensemble corresponding to the state at the output has the following form:

$$\{\{1-p, |\psi\rangle\}, \{p, X|\psi\rangle\}\}, \tag{2}$$

and the density operator of the resulting state is

$$(1-p)|\psi\rangle\langle\psi| + pX|\psi\rangle\langle\psi|X. \tag{3}$$

The following state is a purification of the above density operator (you should quickly check that this relation holds):

$$\sqrt{1-p}|\psi\rangle_A|0\rangle_E + \sqrt{p}X|\psi\rangle_A|1\rangle_E. \tag{4}$$

We label the original system as A and label the purification system as E. In this context, we can view the purification system as the environment of the channel.

There is another way for interpreting the dynamics of the above bit-flip channel. Instead of determining the ensemble for the channel and then purifying, we can say that the channel directly implements the following map from the system A to the larger joint system AE:

$$|\psi\rangle_A \to \sqrt{1-p}\,|\psi\rangle_A\,|0\rangle_E + \sqrt{p}X|\psi\rangle_A|1\rangle_E. \tag{5}$$

We see that any $p \in (0,1)$, i.e., any amount of noise in the channel, can lead to entanglement of the input system with the environment E. We then obtain the noisy dynamics of the channel by discarding (tracing out) the environment system E.

Exercise 1. *Find two input states for which the map in* (5) *does not lead to entanglement between systems A and E.*

The map in (5) is an *isometric extension* of the bit-flip channel. Let us label it as $U_{A\to AE}$ where the notation indicates that the input system is A and the output system is AE. As discussed before, an isometry is similar to a unitary operator but different because it maps states in one Hilbert space (for an input system) to states in a larger Hilbert space (which could be for a joint system). It generally does not admit a square matrix representation, but instead admits a rectangular matrix representation. The matrix representation of the isometric operation in (5) consists of the following matrix elements:

$$\begin{bmatrix} \langle 0|_A\langle 0|_E U_{A\to AE}|0\rangle_A & \langle 0|_A\langle 0|_E U_{A\to AE}|1\rangle_A \\ \langle 0|_A\langle 1|_E U_{A\to AE}|0\rangle_A & \langle 0|_A\langle 1|_E U_{A\to AE}|1\rangle_A \\ \langle 1|_A\langle 0|_E U_{A\to AE}|0\rangle_A & \langle 1|_A\langle 0|_E U_{A\to AE}|1\rangle_A \\ \langle 1|_A\langle 1|_E U_{A\to AE}|0\rangle_A & \langle 1|_A\langle 1|_E U_{A\to AE}|1\rangle_A \end{bmatrix} = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{p} \\ 0 & \sqrt{1-p} \\ \sqrt{p} & 0 \end{bmatrix}. \tag{6}$$

There is no reason that we have to choose the environment states as we did in (5). We could have chosen the environment states to be any orthonormal basis—isometric behavior only requires that the states on the environment be distinguishable. This is related to the fact that all purifications are related by an isometry acting on the purifying system.

2.1.1 An Isometry is Part of a Unitary on a Larger System

We can view the dynamics in (5) as an interaction between an initially pure environment and the qubit state $|\psi\rangle$. So, an equivalent way to implement an isometric mapping is with a two-step procedure. We first assume that the environment of the channel is in a pure state $|0\rangle_E$ before the interaction begins. The joint state of the qubit $|\psi\rangle$ and the environment is

$$|\psi\rangle_A|0\rangle_E. \tag{7}$$

These two systems then interact according to a unitary operator V_{AE}. We can specify two columns of the unitary operator (we make this more clear in a bit) by means of the isometric mapping in (5):

$$V_{AE}|\psi\rangle_A|0\rangle_E = \sqrt{1-p}\,|\psi\rangle_A\,|0\rangle_E + \sqrt{p}X|\psi\rangle_A|1\rangle_E. \tag{8}$$

In order to specify the full unitary V_{AE}, we must also specify how the map behaves when the initial state of the qubit and the environment is

$$|\psi\rangle_A|1\rangle_E. \tag{9}$$

We choose the mapping to be as follows so that the overall interaction is unitary:

$$V_{AE}|\psi\rangle_A|1\rangle_E = \sqrt{p}\,|\psi\rangle_A\,|0\rangle_E - \sqrt{1-p}X\,|\psi\rangle_A\,|1\rangle_E. \tag{10}$$

2.1.2 Complementary Channel

We may not only be interested in the receiver's output of the quantum channel. We may also be interested in determining the environment's output from the channel. This idea becomes increasingly important as we proceed in our study of quantum Shannon theory. We should consider all parties in a quantum protocol, and the purified quantum theory allows us to do so. We consider the environment as one of the parties in a quantum protocol because the environment could also be receiving some quantum information from the sender.

We can obtain the environment's output from the quantum channel simply by tracing out every system besides the environment. The map from the sender to the environment is known as a *complementary channel*. In our example of the isometric extension of the bit-flip channel in (5), we can check that the environment receives the following output state if the channel input is $|\psi\rangle_A$:

$$\begin{aligned}
&\mathrm{Tr}_A\left\{\left(\sqrt{1-p}\,|\psi\rangle_A\,|0\rangle_E+\sqrt{p}X|\psi\rangle_A|1\rangle_E\right)\left(\sqrt{1-p}\langle\psi|_A\langle 0|_E+\sqrt{p}\langle\psi|_A X\langle 1|_E\right)\right\}\\
&=\mathrm{Tr}_A\left\{(1-p)|\psi\rangle\langle\psi|_A\otimes|0\rangle\langle 0|_E+\sqrt{p(1-p)}X|\psi\rangle\langle\psi|_A\otimes|1\rangle\langle 0|_E\right\}\\
&\qquad+\mathrm{Tr}_A\left\{\sqrt{p\,(1-p)}|\psi\rangle\langle\psi|_A X\otimes|0\rangle\langle 1|_E+pX|\psi\rangle\langle\psi|_A X\otimes|1\rangle\langle 1|_E\right\} &(11)\\
&=(1-p)|0\rangle\langle 0|_E+\sqrt{p\,(1-p)}\langle\psi|X|\psi\rangle|1\rangle\langle 0|_E\\
&\qquad+\sqrt{p(1-p)}\langle\psi|X|\psi\rangle|0\rangle\langle 1|_E+p|1\rangle\langle 1|_E &(12)\\
&=(1-p)|0\rangle\langle 0|_E+\sqrt{p\,(1-p)}\langle\psi|X|\psi\rangle\left(|1\rangle\langle 0|_E+|0\rangle\langle 1|_E\right)+p|1\rangle\langle 1|_E &(13)\\
&=(1-p)|0\rangle\langle 0|_E+\sqrt{p\,(1-p)}2\,\mathrm{Re}\left\{\alpha^*\beta\right\}\left(|1\rangle\langle 0|_E+|0\rangle\langle 1|_E\right)+p|1\rangle\langle 1|_E, &(14)
\end{aligned}$$

where in the last line we assume that the qubit $|\psi\rangle\equiv\alpha|0\rangle+\beta|1\rangle$.

It is helpful to examine several cases of the above example. Consider the case in which the noise parameter $p=0$ or $p=1$. In this case, the environment receives one of the respective states $|0\rangle$ or $|1\rangle$. Therefore, in these cases, the environment does not receive any of the quantum information about the state $|\psi\rangle$ transmitted down the channel—it does not learn anything about the probability amplitudes α or β. This viewpoint is a completely different way to see that the channel is truly noiseless in these cases. A channel is noiseless if the environment of the channel does not learn anything about the states that we transmit through it, i.e., the channel does not leak quantum information to the environment. Now let us consider the case in which $p\in(0,1)$. As p approaches $1/2$ from either above or below, the amplitude $\sqrt{p(1-p)}$ of the off-diagonal terms is a monotonic function that reaches its peak at $1/2$. Thus, at the peak $1/2$, the off-diagonal terms are the strongest, implying that the environment is generally "stealing" much of the coherence from the original quantum state $|\psi\rangle$.

Exercise 2. *Show that the receiver's output density operator for a bit-flip channel with* $p=1/2$ *is the same as what the environment obtains.*

2.2 Isometric Extension of a Quantum Channel

We now give a general definition for an isometric extension of a quantum channel:

Definition 3 (Isometric Extension)**.** *Let* $\mathcal{H}_A$ *and* $\mathcal{H}_B$ *be Hilbert spaces, and let* $\mathcal{N}:\mathcal{B}(\mathcal{H}_A)\to\mathcal{B}(\mathcal{H}_B)$ *be a quantum channel. Let* $\mathcal{H}_E$ *be a Hilbert space with dimension no smaller than the Choi*

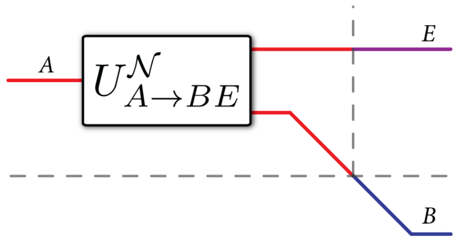

Figure 1: This figure depicts an isometric extension $U^{\mathcal{N}}_{A\to BE}$ of a quantum channel $\mathcal{N}_{A\to B}$. The extension $U^{\mathcal{N}}_{A\to BE}$ includes the inaccessible environment on system E as a "receiver" of quantum information. Ignoring the environment E gives the quantum channel $\mathcal{N}_{A\to B}$.

rank of the channel $\mathcal{N}$. An isometric extension or Stinespring dilation $U : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ of the channel $\mathcal{N}$ is a linear isometry such that

$$\mathrm{Tr}_E\{UX_AU^\dagger\} = \mathcal{N}_{A\to B}(X_A), \tag{15}$$

for $X_A \in \mathcal{B}(\mathcal{H}_A)$. The fact that U is an isometry is equivalent to the following conditions:

$$U^\dagger U = I_A, \qquad UU^\dagger = \Pi_{BE}, \tag{16}$$

where Π_{BE} is a projection of the tensor-product Hilbert space $\mathcal{H}_B \otimes \mathcal{H}_E$.

Notation 4. *We often write a channel $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ as $\mathcal{N}_{A\to B}$ in order to indicate the input and output systems explicitly. Similarly, we often write an isometric extension $U : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ of $\mathcal{N}$ as $U^{\mathcal{N}}_{A\to BE}$ in order to indicate its association with $\mathcal{N}$ explicitly, as well the fact that it accepts an input system A and has output systems B and E. The system E is often referred to as an "environment" system. Finally, there is a quantum channel $\mathcal{U}^{\mathcal{N}}_{A\to BE}$ associated to an isometric extension $U^{\mathcal{N}}_{A\to BE}$, which is defined by*

$$\mathcal{U}^{\mathcal{N}}_{A\to BE}(X_A) = UX_AU^\dagger, \tag{17}$$

for $X_A \in \mathcal{B}(\mathcal{H}_A)$. Note that $\mathcal{U}^{\mathcal{N}}_{A\to BE}$ is a quantum channel with a single Kraus operator U given that $U^\dagger U = I_A$.

We can think of an isometric extension of a quantum channel as a purification of that channel: the environment system E is analogous to the purification system from Section **??** because we trace over it to get back the original channel. An isometric extension *extends* the original channel because it produces the evolution of the quantum channel $\mathcal{N}_{A\to B}$ if we trace out the environment system E. It also behaves as an *isometry*—it is analogous to a rectangular matrix that behaves somewhat like a unitary operator. The matrix representation of an isometry is a rectangular matrix formed from selecting only a few of the columns from a unitary matrix. The property $U^\dagger U = I_A$ indicates that the isometry behaves analogously to a unitary operator, because we can determine an inverse operation simply by taking its conjugate transpose. The property $UU^\dagger = \Pi_{BE}$ distinguishes an isometric operation from a unitary one. It states that the isometry takes states in the input system A to a particular subspace of the joint system BE. The projector Π_{BE} projects onto the subspace where the isometry takes input quantum states. Figure 1 depicts a quantum circuit for an isometric extension.

2.2.1 Isometric Extension from Kraus Operators

It is possible to determine an isometric extension of a quantum channel directly from a set of Kraus operators. Consider a quantum channel $\mathcal{N}_{A\to B}$ with the following Kraus representation:

$$\mathcal{N}_{A\to B}(\rho_A) = \sum_j N_j \rho_A N_j^\dagger. \tag{18}$$

An isometric extension of the channel $\mathcal{N}_{A\to B}$ is the following linear map:

$$U^{\mathcal{N}}_{A\to BE} \equiv \sum_j N_j \otimes |j\rangle_E \,. \tag{19}$$

It is straightforward to verify that the above map is an isometry:

$$\begin{aligned}
\left(U^{\mathcal{N}}\right)^\dagger U^{\mathcal{N}} &= \left(\sum_k N_k^\dagger \otimes \langle k|_E\right)\left(\sum_j N_j \otimes |j\rangle_E\right) && (20)\\
&= \sum_{k,j} N_k^\dagger N_j \langle k|j\rangle && (21)\\
&= \sum_k N_k^\dagger N_k && (22)\\
&= I_A. && (23)
\end{aligned}$$

The last equality follows from the completeness condition of the Kraus operators. As a consequence, we get that $U^{\mathcal{N}}\left(U^{\mathcal{N}}\right)^\dagger$ is a projector on the joint system BE, which follows because $(UU^\dagger)(UU^\dagger) = U(U^\dagger U)U^\dagger = UIU^\dagger = UU^\dagger$. Finally, we should verify that $U^{\mathcal{N}}$ is an extension of $\mathcal{N}$. Applying the channel $\mathcal{U}^{\mathcal{N}}_{A\to BE}$ to an arbitrary density operator ρ_A gives the following map:

$$\begin{aligned}
\mathcal{U}^{\mathcal{N}}_{A\to BE}(\rho_A) &\equiv U^{\mathcal{N}}\rho_A\left(U^{\mathcal{N}}\right)^\dagger && (24)\\
&= \left(\sum_j N_j \otimes |j\rangle_E\right)\rho_A\left(\sum_k N_k^\dagger \otimes \langle k|_E\right) && (25)\\
&= \sum_{j,k} N_j \rho_A N_k^\dagger \otimes |j\rangle\langle k|_E, && (26)
\end{aligned}$$

and tracing out the environment system gives back the original quantum channel $\mathcal{N}_{A\to B}$:

$$\begin{aligned}
\mathrm{Tr}_E\left\{\mathcal{U}^{\mathcal{N}}_{A\to BE}(\rho_A)\right\} &= \sum_j N_j \rho_A N_j^\dagger && (27)\\
&= \mathcal{N}_{A\to B}(\rho_A). && (28)
\end{aligned}$$

Exercise 5. *Show that all isometric extensions of a quantum channel are equivalent up to an isometry on the environment system.*

2.2.2 Complementary Channel

In the purified quantum theory, it is useful to consider all parties that are participating in a given protocol. One such party is the environment of the channel, even if it is not necessarily an active participant in a protocol. However, in a cryptographic setting, in some sense the environment is active, and we associate it with an eavesdropper, thus personifying it as "Eve."

For any quantum channel $\mathcal{N}_{A\to B}$, there exists an isometric extension $U^{\mathcal{N}}_{A\to BE}$ of that channel. The complementary channel $\mathcal{N}^c_{A\to E}$ is a quantum channel from the sender to the environment, formally defined as follows:

Definition 6 (Complementary Channel)**.** *Let $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a quantum channel, and let $U : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ be an isometric extension of the channel $\mathcal{N}$. The complementary channel $\mathcal{N}^c : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_E)$ of $\mathcal{N}$, associated with U, is defined as follows:*

$$\mathcal{N}^c(X_A) = \mathrm{Tr}_B\left\{UX_AU^\dagger\right\}, \tag{29}$$

for $X_A \in \mathcal{B}(\mathcal{H}_A)$.

That is, we obtain a complementary channel by tracing out Bob's system B from the output of an isometric extension. It captures the noise that Eve "sees" by having her system coupled to Bob's system.

Exercise 7. *Show that Eve's density operator (the output of a complementary channel) is of the following form:*

$$\rho \to \sum_{i,j} \mathrm{Tr}\{N_i\rho N_j^\dagger\}|i\rangle\langle j|, \tag{30}$$

if we take an isometric extension of the channel to be of the form in (19)*.*

The complementary channel is unique only up to an isometry acting on Eve's system. It inherits this property from the fact that an isometric extension of a quantum channel is unique only up to isometries acting on Eve's system. For all practical purposes, this lack of uniqueness does not affect our study of the noise that Eve sees because the measures of noise that we use later are invariant with respect to isometries acting on Eve's system.

2.3 Further Examples of Isometric Extensions

2.3.1 Generalized Dephasing Channels

A generalized dephasing channel is one that preserves states diagonal in some preferred orthonormal basis $\{|x\rangle\}$, but it can add arbitrary phases to the off-diagonal elements of a density operator represented in this basis. An isometric extension of a generalized dephasing channel acts as follows on the basis $\{|x\rangle\}$:

$$U^{\mathcal{N}_\mathrm{D}}_{A'\to BE}\,|x\rangle_{A'} = |x\rangle_B|\varphi_x\rangle_E, \tag{31}$$

where $|\varphi_x\rangle_E$ is some state for the environment (these states need not be mutually orthogonal). Thus, we can represent the isometry as follows:

$$U^{\mathcal{N}_\mathrm{D}}_{A'\to BE} \equiv \sum_x |x\rangle_B|\varphi_x\rangle_E\langle x|_{A'}, \tag{32}$$

and its action on a density operator ρ is

$$U^{\mathcal{N}_\mathrm{D}}\rho\left(U^{\mathcal{N}_\mathrm{D}}\right)^\dagger = \sum_{x,x'}\langle x|\rho|x'\rangle \quad |x\rangle\langle x'|_B \otimes |\varphi_x\rangle\langle\varphi_{x'}|_E. \tag{33}$$

Tracing out the environment gives the action of the channel $\mathcal{N}_\text{D}$ to the receiver

$$\mathcal{N}_\text{D}(\rho) = \sum_{x,x'} \langle x|\rho|x'\rangle \langle \varphi_{x'}|\varphi_x\rangle \ \ |x\rangle\langle x'|_B, \tag{34}$$

where we observe that this channel preserves the diagonal components $\{|x\rangle\langle x|\}$ of ρ, but it multiplies the $d\,(d-1)$ off-diagonal elements of ρ by arbitrary phases, depending on the $d\,(d-1)$ overlaps $\langle \varphi_{x'}|\varphi_x\rangle$ of the environment states (where $x \neq x'$). Tracing out the receiver gives the action of the complementary channel $\mathcal{N}_\text{D}^c$ to the environment

$$\mathcal{N}_\text{D}^c(\rho) = \sum_x \langle x|\rho|x\rangle \ \ |\varphi_x\rangle\langle\varphi_x|_E. \tag{35}$$

Observe that the channel to the environment is entanglement-breaking. That is, the action of the channel is the same as first performing a von Neumann measurement in the basis $\{|x\rangle\}$ and preparing a state $|\varphi_x\rangle_E$ conditioned on the outcome of the measurement (it is a classical–quantum channel, as discussed in Section **??**). Additionally, the receiver Bob can simulate the action of this channel to the receiver by performing the same actions on the state that he receives.

Exercise 8. *Explicitly show that the following qubit dephasing channel is a special case of a generalized dephasing channel:*

$$\rho \to (1-p)\rho + pZ\rho Z. \tag{36}$$

2.3.2 Quantum Hadamard Channels

Quantum Hadamard channels are those whose complements are entanglement-breaking, and so generalized dephasing channels are a subclass of quantum Hadamard channels. We can write the output of a quantum Hadamard channel as the Hadamard product (element-wise multiplication) of a representation of the input density operator with another operator. To discuss how this comes about, suppose that the complementary channel $\mathcal{N}^c_{A\to E}$ of a channel $\mathcal{N}_{A\to B}$ is entanglement-breaking. Then, using the fact that its Kraus operators $|\xi_i\rangle_E \langle\zeta_i|_A$ are unit rank (see Theorem **??**) and the construction in (19) for an isometric extension, we can write an isometric extension $U^{\mathcal{N}^c}$ for $\mathcal{N}^c$ as

$$U^{\mathcal{N}^c}\rho_A \left(U^{\mathcal{N}^c}\right)^\dagger = \sum_{i,j} |\xi_i\rangle_E \langle\zeta_i|_A\, \rho_A \,|\zeta_j\rangle_A \langle\xi_j|_E \otimes |i\rangle_B \langle j|_B \tag{37}$$

$$= \sum_{i,j} \langle\zeta_i|_A\, \rho_A \,|\zeta_j\rangle_A\, |\xi_i\rangle_E \langle\xi_j|_E \otimes |i\rangle_B \langle j|_B\,. \tag{38}$$

The sets $\{|\xi_i\rangle_E\}$ and $\{|\zeta_i\rangle_A\}$ each do not necessarily consist of orthonormal states, but the set $\{|i\rangle_B\}$ does because it is the environment of the complementary channel. Tracing over the system E gives the original channel from system A to B:

$$\mathcal{N}^\text{H}_{A\to B}(\rho_A) = \sum_{i,j} \langle\zeta_i|_A\, \rho_A\, |\zeta_j\rangle_A \langle\xi_j|\xi_i\rangle_E\, |i\rangle_B\langle j|_B. \tag{39}$$

Let Σ denote the matrix with elements $[\Sigma]_{i,j} = \langle\zeta_i|_A\,\rho_A\,|\zeta_j\rangle_A$, a representation of the input state ρ, and let Γ denote the matrix with elements $[\Gamma]_{i,j} = \langle\xi_i|\xi_j\rangle_E$. Then, from (39), it is clear that the output of the channel is the Hadamard product $*$ of Σ and $\Gamma^\dagger$ with respect to the basis $\{|i\rangle_B\}$:

$$\mathcal{N}^\text{H}_{A\to B}(\rho) = \Sigma * \Gamma^\dagger. \tag{40}$$

For this reason, such a channel is known as a Hadamard channel.

Hadamard channels are *degradable*, as introduced in the following definition:

Definition 9 (Degradable Channel)**.** *Let $\mathcal{N}_{A\to B}$ be a quantum channel, and let $\mathcal{N}^c_{A\to E}$ denote a complementary channel for $\mathcal{N}_{A\to B}$. The channel $\mathcal{N}_{A\to B}$ is degradable if there exists a degrading channel $\mathcal{D}_{B\to E}$ such that*

$$\mathcal{D}_{B\to E}(\mathcal{N}_{A\to B}(X_A)) = \mathcal{N}^c_{A\to E}(X_A), \tag{41}$$

for all $X_A \in \mathcal{B}(\mathcal{H}_A)$.

To see that a quantum Hadamard channel is degradable, let Bob perform a von Neumann measurement of his state in the basis $\{|i\rangle_B\}$ and prepare the state $|\xi_i\rangle_E$ conditioned on the outcome of the measurement. This procedure simulates the complementary channel $\mathcal{N}^c_{A\to E}$ and also implies that the degrading map $\mathcal{D}_{B\to E}$ is entanglement-breaking. To be more precise, the Kraus operators of the degrading map $\mathcal{D}_{B\to E}$ are $\{|\xi_i\rangle_E \langle i|_B\}$ so that

$$\mathcal{D}_{B\to E}(\mathcal{N}^{\mathrm{H}}_{A\to B}(\sigma_A)) = \sum_i |\xi_i\rangle_E \langle i|_B \mathcal{N}_{A\to B}(\sigma_A)|i\rangle_B \langle \xi_i|_E \tag{42}$$

$$= \sum_i \langle i|_A \sigma_A |i\rangle_A |\xi_i\rangle \langle \xi_i|_E\,, \tag{43}$$

demonstrating that this degrading map effectively simulates the complementary channel $\mathcal{N}^{\mathrm{H}}_{A\to E}$. Note that we can view this degrading map as the composition of two maps: a first map $\mathcal{D}^1_{B\to Y}$ performs the von Neumann measurement, leading to a classical variable Y, and a second map $\mathcal{D}^2_{Y\to E}$ performs the state preparation, conditional on the value of the classical variable Y. We can therefore write $\mathcal{D}_{B\to E} = \mathcal{D}^2_{Y\to E} \circ \mathcal{D}^1_{B\to Y}$. This particular form of the channel has implications for its quantum capacity and its more general capacities. Observe that a generalized dephasing channel from the previous section is a quantum Hadamard channel because the map to its environment is entanglement-breaking.

2.4 Isometric Extension and the Adjoint of a Quantum Channel

Recall the notion of an adjoint of a quantum channel from before. Here we show an alternate way of representing an adjoint of a quantum channel using an isometric extension of it.

Proposition 10. *Let $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a quantum channel and let $U : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ be an isometric extension of it. Then the adjoint map $\mathcal{N}^\dagger : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ can be written as follows:*

$$\mathcal{N}^\dagger(Y_B) = U^\dagger(Y_B \otimes I_E)U, \tag{44}$$

for $Y_B \in \mathcal{B}(\mathcal{H}_B)$.

Proof. We can see this by using the definition of the adjoint map, the definition of an isometric extension (Definition 3), and the definition of partial trace. Consider from the definition of the adjoint map that $\mathcal{N}^\dagger$ is such that

$$\langle Y_B, \mathcal{N}(X_A)\rangle = \left\langle \mathcal{N}^\dagger(Y_B), X_A \right\rangle, \tag{45}$$

for all $X_A \in \mathcal{B}(\mathcal{H}_A)$ and $Y_B \in \mathcal{B}(\mathcal{H}_B)$. Then

$$\langle Y_B, \mathcal{N}(X_A)\rangle = \mathrm{Tr}\{Y_B^\dagger \mathcal{N}(X_A)\} \tag{46}$$
$$= \mathrm{Tr}\{Y_B^\dagger \,\mathrm{Tr}_E\{U X_A U^\dagger\}\} \tag{47}$$
$$= \mathrm{Tr}\{(Y_B^\dagger \otimes I_E) U X_A U^\dagger\} \tag{48}$$
$$= \mathrm{Tr}\{U^\dagger (Y_B^\dagger \otimes I_E) U X_A\} \tag{49}$$
$$= \mathrm{Tr}\{\left[U^\dagger (Y_B \otimes I_E) U\right]^\dagger X_A\} \tag{50}$$
$$= \left\langle U^\dagger \left(Y_B \otimes I_E\right) U, X_A \right\rangle. \tag{51}$$

The second equality is from the definition of an isometric extension. The third equality follows by applying the definition of partial trace. The fourth uses cyclicity of trace. Since we have shown that $\langle Y_B, \mathcal{N}(X_A)\rangle = \left\langle U^\dagger \left(Y_B \otimes I_E\right) U, X_A \right\rangle$ for all $X_A \in \mathcal{B}(\mathcal{H}_A)$ and $Y_B \in \mathcal{B}(\mathcal{H}_B)$, the statement in (44) follows. □

We can verify the formula in (44) in a different way. Suppose that we have a Kraus representation of the channel $\mathcal{N}$ as follows:

$$\mathcal{N}(X_A) = \sum_l V_l X_A V_l^\dagger, \tag{52}$$

where $V_l \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_B)$ for all l and $\sum_l V_l^\dagger V_l = I_A$. An isometric extension U for this channel is then as given in (19):

$$U = \sum_l V_l \otimes |l\rangle_E, \tag{53}$$

where $\{|l\rangle_E\}$ is some orthonormal basis. We can then explicitly compute the formula in (44) as follows:

$$U^\dagger (Y_B \otimes I_E) U = \left(\sum_l V_l^\dagger \otimes \langle l|_E\right) (Y_B \otimes I_E) \left(\sum_{l'} V_{l'} \otimes |l'\rangle_E\right) \tag{54}$$
$$= \sum_{l,l'} V_l^\dagger Y_B V_{l'} \langle l|l'\rangle_E \tag{55}$$
$$= \sum_l V_l^\dagger Y_B V_l \tag{56}$$
$$= \mathcal{N}^\dagger(Y_B), \tag{57}$$

where the last equality follows from what we calculated before.

3 Coherent Measurement

We end this chapter by discussing a coherent measurement. This last section, combined with the notion of an isometric extension of a quantum channel, shows that it is sufficient to describe all of the quantum theory in the so-called "traditionalist" way by using only unitary evolutions and von Neumann projective measurements.

Suppose that we have a set of measurement operators $\{M_j\}_j$ such that $\sum_j M_j^\dagger M_j = I$. In the noisy quantum theory, we found that the post-measurement state of a measurement on a quantum system S with density operator ρ is

$$\frac{M_j \rho M_j^\dagger}{p_J(j)}, \tag{58}$$

where the measurement outcome j occurs with probability

$$p_J(j) = \text{Tr}\left\{M_j^\dagger M_j \rho\right\}. \tag{59}$$

We would like a way to perform the above measurement on system S in a *coherent* fashion. The isometry in (19) gives a hint for how we can structure such a coherent measurement. We can build the coherent measurement as the following isometry:

$$U_{S\to SS'} \equiv \sum_j M_S^j \otimes |j\rangle_{S'}. \tag{60}$$

Appying this isometry to a density operator ρ_S gives the following state:

$$\mathcal{U}_{S\to SS'}(\rho_S) = U_{S\to SS'}\rho_S (U_{S\to SS'})^\dagger \tag{61}$$

$$= \sum_{j,j'} M_S^j \rho_S (M_S^{j'})^\dagger \otimes |j\rangle \langle j'|_{S'}. \tag{62}$$

We can then apply a von Neumann measurement with projection operators $\{|j\rangle\langle j|\}_j$ to the system S', which gives the following post-measurement state:

$$\frac{(I_S \otimes |j\rangle\langle j|_{S'})(\mathcal{U}_{S\to SS'}(\rho_S))(I_S \otimes |j\rangle \langle j|_{S'})}{\text{Tr}\left\{(I_S \otimes |j\rangle\langle j|_{S'})(U_{S\to SS'}(\rho_S))\right\}} = \frac{M_S^j \rho_S (M_S^j)^\dagger}{\text{Tr}\left\{(M_S^j)^\dagger M_S^j \rho_S\right\}} \otimes |j\rangle \langle j|_{S'}. \tag{63}$$

The result is then the same as that in (58). In fact, this is the same as the way in which we previously motivated an alternate description of quantum measurements.