**PHYS 7895: Quantum Information Theory** Fall 2015

# Lecture 11 — September 30, 2015

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## 1 Overview

In the last lecture, we discussed how to combine channels together and how preparations and measurements can be viewed as quantum channels.

This lecture begins our first exciting application of the postulates of the quantum theory to quantum communication. We study the fundamental, unit quantum communication protocols. These protocols involve a single sender Alice and a single receiver Bob. The protocols are ideal and noiseless because we assume that Alice and Bob can exploit perfect classical communication, perfect quantum communication, and perfect entanglement. At the end of this chapter, we suggest how to incorporate imperfections into these protocols for later study.

Alice and Bob may wish to perform one of several quantum information-processing tasks, such as the transmission of classical information, quantum information, or entanglement. Several fundamental protocols make use of these resources:

1. We will see that noiseless entanglement is an important resource in quantum Shannon theory because it enables Alice and Bob to perform other protocols that are not possible with classical resources only. We will present a simple, idealized protocol for generating entanglement, named *entanglement distribution.*

2. Alice may wish to communicate classical information to Bob. A trivial method, named *elementary coding*, is a simple way for doing so and we discuss it briefly.

3. A more interesting technique for transmitting classical information is *super-dense coding.* It exploits a noiseless qubit channel and shared entanglement to transmit more classical information than would be possible with a noiseless qubit channel alone.

4. Finally, Alice may wish to transmit quantum information to Bob. A trivial method for her to do so to exploit a noiseless qubit channel. However, it is useful to have other ways for transmitting quantum information because such a resource is difficult to engineer in practice. An alternative, surprising method for transmitting quantum information is *quantum teleportation.* The teleportation protocol exploits classical communication and shared entanglement to transmit quantum information.

Each of these protocols is a fundamental unit protocol and provides a foundation for asking further questions in quantum Shannon theory. In fact, the discovery of these latter two protocols was the stimulus for much of the original research in quantum Shannon theory. One could take each of these protocols and ask about its performance if one or more of the resources involved is noisy rather than noiseless. Later chapters of this book explore many of these possibilities.

This chapter introduces the technique of *resource counting*, which is of practical importance because it quantifies the communication cost of achieving a certain task. We include only non-local resources in a resource count—non-local resources include classical or quantum communication or shared entanglement.

It is important to minimize the use of certain resources, such as noiseless entanglement or a noiseless qubit channel, in a given protocol because they are expensive. Given a certain implementation of a quantum information-processing task, we may wonder if there is a way of implementing it that consumes fewer resources. A proof that a given protocol is the best that we can hope to do is an optimality proof. We argue, based on good physical grounds, that the protocols in this chapter are the best implementations of the desired quantum information-processing task. Later we give information-theoretic proofs of optimality.

# 2 Non-local Unit Resources

We first briefly define what we mean by a noiseless qubit channel, a noiseless classical bit channel, and noiseless entanglement. Each of these resources is a *non-local, unit resource.* A resource is *non-local* if two spatially separated parties share it or if one party uses it to communicate to another. We say that a resource is *unit* if it comes in some "gold standard" form, such as qubits, classical bits, or entangled bits. It is important to establish these definitions so that we can check whether a given protocol is truly simulating one of these resources.

A noiseless qubit channel is any mechanism that implements the following map:

$$|i\rangle_A \to |i\rangle_B, \tag{1}$$

extended linearly to arbitrary state vectors and where $i \in \{0,1\}$, $\{|0\rangle_A, |1\rangle_A\}$ is some preferred orthonormal basis on Alice's system, and $\{|0\rangle_B, |1\rangle_B\}$ is some preferred orthonormal basis on Bob's system. The bases do not have to be the same, but it must be clear which basis each party is using. The above map is linear so that it preserves arbitrary superposition states (it preserves any qubit). For example, the map acts as follows on a superposition state:

$$\alpha|0\rangle_A + \beta|1\rangle_A \to \alpha|0\rangle_B + \beta|1\rangle_B. \tag{2}$$

We can also write it as the following isometry:

$$\sum_{i=0}^{1} |i\rangle_B \langle i|_A. \tag{3}$$

Any information-processing protocol that implements the above map simulates a noiseless qubit channel. We label the communication resource of a noiseless qubit channel as follows:

$$[q \to q], \tag{4}$$

where the notation indicates one forward use of a noiseless qubit channel.

A noiseless classical bit channel is any mechanism that implements the following map:

$$|i\rangle\langle i|_A \to |i\rangle\langle i|_B, \tag{5}$$
$$|i\rangle\langle j|_A \to 0 \quad \text{for } i \neq j, \tag{6}$$

extended linearly to density operators and where $i, j \in \{0, 1\}$ and the orthonormal bases are again arbitrary. This channel maintains the diagonal elements of a density operator in the basis $\{|0\rangle_A, |1\rangle_A\}$, but it eliminates the off-diagonal elements. We can write it as the following linear map acting on a density operator $\rho_A$:

$$\rho_A \to \sum_{i=0}^{1} |i\rangle_B \langle i|_A \rho_A |i\rangle_A \langle i|_B. \tag{7}$$

The form above is consistent with our former definition for noiseless classical channels. This resource is weaker than a noiseless qubit channel because it does not require Alice and Bob to maintain arbitrary superposition states—it merely transfers classical information. Alice can use the above channel to transmit classical information to Bob. She can prepare either of the classical states $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$, send it through the classical channel, and Bob performs a computational basis measurement to determine the message Alice transmits. We denote the communication resource of a noiseless classical bit channel as follows:

$$[c \to c], \tag{8}$$

where the notation indicates one forward use of a noiseless classical bit channel.

We can study other ways of transmitting classical information. For example, suppose that Alice flips a fair coin that chooses the state $|0\rangle_A$ or $|1\rangle_A$ with equal probability. The resulting state is the following density operator:

$$\frac{1}{2}\left(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A\right). \tag{9}$$

Suppose that she sends the above state through a noiseless classical channel. The resulting density operator for Bob is as follows:

$$\frac{1}{2}\left(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B\right). \tag{10}$$

The above classical bit channel map does not preserve off-diagonal elements of a density operator. Suppose instead that Alice prepares a superposition state

$$\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}. \tag{11}$$

The density operator corresponding to this state is

$$\frac{1}{2}\left(|0\rangle\langle 0|_A + |0\rangle\langle 1|_A + |1\rangle\langle 0|_A + |1\rangle\langle 1|_A\right). \tag{12}$$

Suppose Alice then transmits this state through the above classical channel. The classical channel eliminates all the off-diagonal elements of the density operator and the resulting state for Bob is as follows:

$$\frac{1}{2}\left(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B\right). \tag{13}$$

Thus, it is impossible for a noiseless classical channel to simulate a noiseless qubit channel because it cannot maintain arbitrary superposition states. However, it is possible for a noiseless qubit channel to simulate a noiseless classical bit channel and we denote this fact with the following *resource inequality*:

$$[q \to q] \geq [c \to c]\,. \tag{14}$$

Noiseless quantum communication is therefore a stronger resource than noiseless classical communication.

The final resource that we consider is shared entanglement. The ebit is our "gold standard" resource for pure bipartite (two-party) entanglement, and we will make this point more clear operationally later on. An ebit is the following state of two qubits:

$$\left|\Phi^+\right\rangle_{AB} \equiv \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right), \tag{15}$$

where Alice possesses the first qubit and Bob possesses the second.

Below, we show how a noiseless qubit channel can generate a noiseless ebit through a simple protocol named *entanglement distribution*. However, an ebit cannot simulate a noiseless qubit channel (for reasons which we explain later). Therefore, noiseless quantum communication is the strongest of all three resources, and entanglement and classical communication are in some sense "orthogonal" to one another because neither can simulate the other.

## 2.1 The Bell States

There are other useful entangled states besides the standard ebit. Suppose that Alice performs a $Z_A$ operation on her share of the ebit $|\Phi^+\rangle_{AB}$. Then the resulting state is

$$\left|\Phi^-\right\rangle_{AB} \equiv \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} - |11\rangle_{AB}\right). \tag{16}$$

Similarly, if Alice performs an $X$ operator or a $Y$ operator, the global state transforms to the following respective states (up to a global phase):

$$\left|\Psi^+\right\rangle_{AB} \equiv \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} + |10\rangle_{AB}\right), \tag{17}$$

$$\left|\Psi^-\right\rangle_{AB} \equiv \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} - |10\rangle_{AB}\right). \tag{18}$$

The states $|\Phi^+\rangle_{AB}$, $|\Phi^-\rangle_{AB}$, $|\Psi^+\rangle_{AB}$, and $|\Psi^-\rangle_{AB}$ are known as the *Bell states* and are the most important entangled states for a two-qubit system. They form an orthonormal basis, called the *Bell basis*, for a two-qubit space. We can also label the Bell states as

$$|\Phi^{zx}\rangle_{AB} \equiv Z_A^z X_A^x \left|\Phi^+\right\rangle_{AB}, \tag{19}$$

where the two-bit binary number $zx$ indicates whether Alice applies $I_A$, $Z_A$, $X_A$, or $Z_A X_A$. Then the states $\left|\Phi^{00}\right\rangle_{AB}$, $\left|\Phi^{01}\right\rangle_{AB}$, $\left|\Phi^{10}\right\rangle_{AB}$, and $\left|\Phi^{11}\right\rangle_{AB}$ are in correspondence with the respective states $|\Phi^+\rangle_{AB}$, $|\Psi^+\rangle_{AB}$, $|\Phi^-\rangle_{AB}$, and $|\Psi^-\rangle_{AB}$.

**Exercise 1.** *Show that the Bell states form an orthonormal basis:*

$$\langle \Phi^{z_1 x_1} | \Phi^{z_2 x_2} \rangle = \delta_{z_1, z_2} \delta_{x_1, x_2}. \tag{20}$$

**Exercise 2.** *Show that the following identities hold:*

$$|00\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left|\Phi^{+}\right\rangle_{AB} + \left|\Phi^{-}\right\rangle_{AB}\right), \tag{21}$$

$$|01\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left|\Psi^{+}\right\rangle_{AB} + \left|\Psi^{-}\right\rangle_{AB}\right), \tag{22}$$

$$|10\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left|\Psi^{+}\right\rangle_{AB} - \left|\Psi^{-}\right\rangle_{AB}\right), \tag{23}$$

$$|11\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\left|\Phi^{+}\right\rangle_{AB} - \left|\Phi^{-}\right\rangle_{AB}\right). \tag{24}$$

# 3 Protocols

## 3.1 Entanglement Distribution

The entanglement distribution protocol is the most basic of the three unit protocols. It exploits one use of a noiseless qubit channel to establish one shared noiseless ebit. It consists of the following two steps:

1. Alice prepares a Bell state locally in her laboratory. She prepares two qubits in the state $|0\rangle_A|0\rangle_{A'}$, where we label the first qubit as $A$ and the second qubit as $A'$. She performs a Hadamard gate on qubit $A$ to produce the following state:

$$\left(\frac{|0\rangle_A + |1\rangle_A}{\sqrt{2}}\right)|0\rangle_{A'}. \tag{25}$$

   She then performs a CNOT gate with qubit $A$ as the source qubit and qubit $A'$ as the target qubit. The state becomes the following Bell state:

$$\left|\Phi^{+}\right\rangle_{AA'} = \frac{|00\rangle_{AA'} + |11\rangle_{AA'}}{\sqrt{2}}. \tag{26}$$

2. She sends qubit $A'$ to Bob with one use of a noiseless qubit channel. Alice and Bob then share the ebit $|\Phi^{+}\rangle_{AB}$.

Figure 1 depicts the entanglement distribution protocol.

The following resource inequality quantifies the non-local resources consumed or generated in the above protocol:

$$[q \to q] \geq [qq], \tag{27}$$

where $[q \to q]$ denotes one forward use of a noiseless qubit channel and $[qq]$ denotes a shared, noiseless ebit. The meaning of the resource inequality is that there exists a protocol that consumes the resource on the left in order to generate the resource on the right. The best analogy is to think of a resource inequality as a "chemical reaction"-like formula, where the protocol is like a chemical reaction that transforms one resource into another.

There are several subtleties to notice about the above protocol and its corresponding resource inequality:

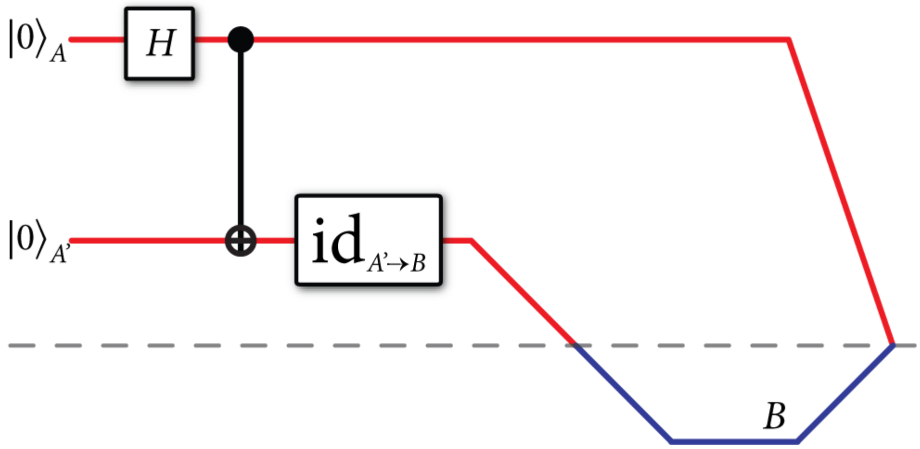


Figure 1: This figure depicts a protocol for entanglement distribution. Alice performs local operations (the Hadamard and CNOT) and consumes one use of a noiseless qubit channel to generate one noiseless ebit $|\Phi^+\rangle_{AB}$ shared with Bob.

1. We are careful with the language when describing the resource state. We described the state $|\Phi^+\rangle$ as a Bell state in the first step because it is a local state in Alice's laboratory. We only used the term "ebit" to describe the state after the second step, when the state becomes a non-local resource shared between Alice and Bob.

2. The resource count involves non-local resources only—we do not factor any local operations, such as the Hadamard gate or the CNOT gate, into the resource count. This line of thinking is different from the theory of computation, where it is of utmost importance to minimize the number of steps involved in a computation. In this book, we are developing a theory of quantum communication and we thus count non-local resources only.

3. We are assuming that it is possible to perform all local operations perfectly. This line of thinking is another departure from practical concerns that one might have in fault-tolerant quantum computation, the study of the propagation of errors in quantum operations. Performing a CNOT gate is a highly nontrivial task at the current stage of experimental development in quantum computation, with most implementations being far from perfect. Nevertheless, we proceed forward with this communication-theoretic line of thinking.

The following exercises outline classical information-processing tasks that are analogous to the task of entanglement distribution.

### 3.1.1 Entanglement and Quantum Communication

Can entanglement enable two parties to communicate quantum information? It is natural to wonder if there is a protocol corresponding to the following resource inequality:

$$[qq] \overset{?}{\geq} [q \to q] \, . \tag{28}$$

Unfortunately, it is physically impossible to construct a protocol that implements the above resource inequality. The argument against such a protocol arises from the theory of relativity. Specifically, the theory of relativity prohibits information transfer or signaling at a speed greater than the speed of light. Suppose that two parties share noiseless entanglement over a large distance. That resource is a static resource, possessing only shared quantum correlations. If a protocol were to exist that implements the above resource inequality, it would imply that two parties could communicate quantum information faster than the speed of light, because they would be exploiting the entanglement for the instantaneous transfer of quantum information.

The entanglement distribution resource inequality is only "one-way," as in (27). Quantum communication is therefore strictly stronger than shared entanglement when no other non-local resources are available.

### 3.2 Elementary Coding

We can also send classical information with a noiseless qubit channel. A simple protocol for doing so is *elementary coding.* This protocol consists of the following steps:

1. Alice prepares either $|0\rangle$ or $|1\rangle$, depending on the classical bit that she would like to send.
2. She transmits this state over the noiseless qubit channel and Bob receives the qubit.
3. Bob performs a measurement in the computational basis to determine the classical bit that Alice transmitted.

Elementary coding succeeds without error because Bob's measurement can always distinguish the classical states $|0\rangle$ and $|1\rangle$. The following resource inequality applies to elementary coding:

$$[q \to q] \geq [c \to c]\,. \tag{29}$$

Again, we are only counting non-local resources in the resource count—we do not count the state preparation at the beginning or the measurement at the end.

If no other resources are available for consumption, the above resource inequality is optimal—one cannot do better than to transmit one classical bit of information per use of a noiseless qubit channel. This result may be a bit frustrating at first, because it may seem that we could exploit the continuous degrees of freedom in the probability amplitudes of a qubit state for encoding more than one classical bit per qubit. Unfortunately, there is no way that we can access the information in the continuous degrees of freedom with any measurement scheme. This protocol is optimal (which one can prove in several ways—this comes later).

### 3.3 Quantum Super-Dense Coding

We now outline a protocol named *super-dense coding.* It is named such because it has the striking property that noiseless entanglement can double the classical communication ability of a noiseless qubit channel. It consists of three steps:

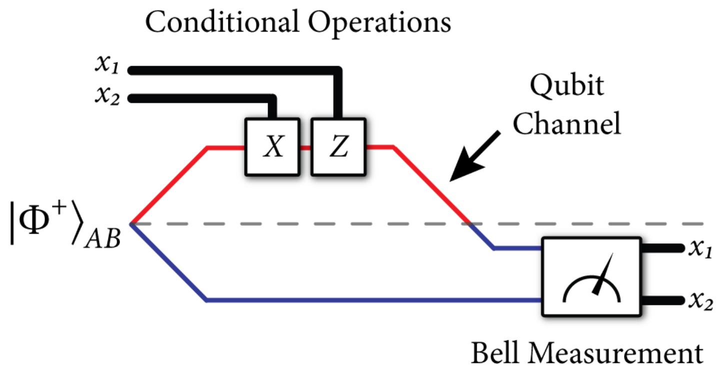


Figure 2: This figure depicts the dense coding protocol. Alice and Bob share an ebit before the protocol begins. Alice would like to transmit two classical bits $x_1 x_2$ to Bob. She performs a Pauli rotation conditional on her two classical bits and sends her half of the ebit over a noiseless qubit channel. Bob can then recover the two classical bits by performing a Bell measurement.

1. Suppose that Alice and Bob share an ebit $|\Phi^+\rangle_{AB}$. Alice applies one of four unitary operations $\{I, X, Z, XZ\}$ to her side of the above state. The state becomes one of the following four Bell states (up to a global phase), depending on the message that Alice chooses:

$$\left|\Phi^+\right\rangle_{AB}, \qquad \left|\Phi^-\right\rangle_{AB}, \qquad \left|\Psi^+\right\rangle_{AB}, \qquad \left|\Psi^-\right\rangle_{AB}. \tag{30}$$

   The definitions of these Bell states are in (16)–(18).

2. She transmits her qubit to Bob with one use of a noiseless qubit channel.

3. Bob performs a Bell measurement (a measurement in the basis $\{|\Phi^+\rangle_{AB}$, $|\Phi^-\rangle_{AB}$, $|\Psi^+\rangle_{AB}$, $|\Psi^-\rangle_{AB}\}$) to distinguish perfectly the four states—he can distinguish the states because they are all orthogonal to each other.

Thus, Alice can transmit two classical bits (corresponding to the four messages) if she shares a noiseless ebit with Bob and uses a noiseless qubit channel. Figure 2 depicts the protocol for quantum super-dense coding.

The super-dense coding protocol implements the following resource inequality:

$$[qq] + [q \to q] \geq 2\,[c \to c]\,. \tag{31}$$

Notice again that the resource inequality counts the use of non-local resources only—we do not count the local operations at the beginning of the protocol or the Bell measurement at the end of the protocol.

Also, notice that we could have implemented two noiseless classical bit channels with two instances of elementary coding:

$$2\,[q \to q] \geq 2\,[c \to c]\,. \tag{32}$$

However, this method is not as powerful as the super-dense coding protocol—in super-dense coding, we consume the weaker resource of an ebit to help transmit two classical bits, instead of consuming the stronger resource of an extra noiseless qubit channel.

The dense coding protocol also transmits the classical bits *privately*. Suppose a third party intercepts the qubit that Alice transmits. There is no measurement that the third party can perform to determine which message Alice transmits because the local density operator of all of the Bell states is the same and equal to the maximally mixed state $\pi_A$ (the information for the eavesdropper is the same irrespective of each message that Alice transmits). The privacy of the protocol is due to Alice and Bob sharing maximal entanglement.

### 3.4 Quantum Teleportation

Perhaps the most striking protocol in noiseless quantum communication is the *quantum teleportation protocol*. The protocol destroys the quantum state of a qubit in one location and recreates it on a qubit at a distant location, with the help of shared entanglement. Thus, the name "teleportation" corresponds well to the mechanism that occurs.

The teleportation protocol is actually a flipped version of the super-dense coding protocol, in the sense that Alice and Bob merely "swap their equipment." The first step in understanding teleportation is to perform a few algebraic steps using the tricks of the tensor product and the Bell state substitutions from Exercise 2. Consider a qubit $|\psi\rangle_{A'}$ that Alice possesses, where

$$|\psi\rangle_{A'} \equiv \alpha|0\rangle_{A'} + \beta|1\rangle_{A'}. \tag{33}$$

Suppose she shares an ebit $|\Phi^+\rangle_{AB}$ with Bob. The joint state of the systems $A'$, $A$, and $B$ is as follows:

$$|\psi\rangle_{A'} \left|\Phi^+\right\rangle_{AB}. \tag{34}$$

Let us first explicitly write out this state:

$$|\psi\rangle_{A'} \left|\Phi^+\right\rangle_{AB} = (\alpha|0\rangle_{A'} + \beta|1\rangle_{A'}) \left(\frac{|00\rangle_{AB} + |11\rangle_{AB}}{\sqrt{2}}\right). \tag{35}$$

Distributing terms gives the following equality:

$$= \frac{1}{\sqrt{2}} \left[\alpha\,|000\rangle_{A'AB} + \beta\,|100\rangle_{A'AB} + \alpha\,|011\rangle_{A'AB} + \beta\,|111\rangle_{A'AB}\right]. \tag{36}$$

We use the relations in Exercise 2 to rewrite the joint system $A'A$ in the Bell basis:

$$= \frac{1}{2}\left[\begin{array}{c} \alpha\,(|\Phi^+\rangle_{A'A} + |\Phi^-\rangle_{A'A})\,|0\rangle_B + \beta\,(|\Psi^+\rangle_{A'A} - |\Psi^-\rangle_{A'A})\,|0\rangle_B \\ +\alpha\,(|\Psi^+\rangle_{A'A} + |\Psi^-\rangle_{A'A})\,|1\rangle_B + \beta\,(|\Phi^+\rangle_{A'A} - |\Phi^-\rangle_{A'A})\,|1\rangle_B \end{array}\right] \tag{37}$$

Simplifying gives the following equivalence:

$$= \frac{1}{2}\left[\begin{array}{c} |\Phi^+\rangle_{A'A}\,(\alpha\,|0\rangle_B + \beta|1\rangle_B) + |\Phi^-\rangle_{A'A}\,(\alpha|0\rangle_B - \beta|1\rangle_B) \\ +\,|\Psi^+\rangle_{A'A}\,(\alpha\,|1\rangle_B + \beta|0\rangle_B) + |\Psi^-\rangle_{A'A}\,(\alpha|1\rangle_B - \beta|0\rangle_B) \end{array}\right]. \tag{38}$$

We can finally rewrite the state as four superposed terms, with a distinct Pauli operator applied to Bob's system $B$ for each term in the superposition:

$$= \frac{1}{2}\left[\left|\Phi^+\right\rangle_{A'A} |\psi\rangle_B + \left|\Phi^-\right\rangle_{A'A} Z|\psi\rangle_B + \left|\Psi^+\right\rangle_{A'A} X|\psi\rangle_B + \left|\Psi^-\right\rangle_{A'A} XZ|\psi\rangle_B\right]. \tag{39}$$

We now outline the three steps of the teleportation protocol (Figure 3 depicts the protocol):

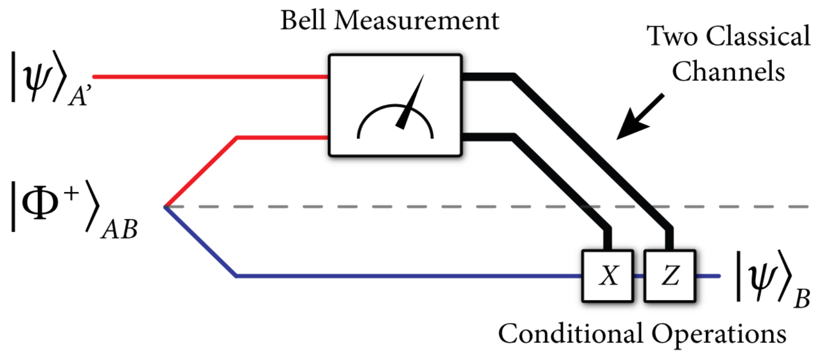


Figure 3: This figure depicts the teleportation protocol. Alice would like to transmit an arbitrary quantum state $|\psi\rangle_{A'}$ to Bob. Alice and Bob share an ebit before the protocol begins. Alice can "teleport" her quantum state to Bob by consuming the entanglement and two uses of a noiseless classical bit channel.

1. Alice performs a Bell measurement on her systems $A'A$. The state collapses to one of the following four states with uniform probability:

$$\left|\Phi^+\right\rangle_{A'A} |\psi\rangle_B, \tag{40}$$

$$\left|\Phi^-\right\rangle_{A'A} Z\, |\psi\rangle_B\,, \tag{41}$$

$$\left|\Psi^+\right\rangle_{A'A} X\, |\psi\rangle_B\,, \tag{42}$$

$$\left|\Psi^-\right\rangle_{A'A} XZ\, |\psi\rangle_B\,. \tag{43}$$

   Notice that the state resulting from the measurement is a product state with respect to the cut $A'A \mid B$, regardless of the outcome of the measurement. At this point, Alice knows whether Bob's state is $|\psi\rangle_B$, $Z|\psi\rangle_B$, $X\,|\psi\rangle_B$, or $XZ|\psi\rangle_B$ because she knows the result of the measurement. On the other hand, Bob does not know anything about the state of his system $B$—his local density operator is the maximally mixed state $\pi_B$ just after Alice performs the measurement. Thus, there is no teleportation of quantum information at this point because Bob's state is completely independent of the original state $|\psi\rangle$. In other words, teleportation cannot be instantaneous.

2. Alice transmits two classical bits to Bob that indicate which of the four measurement results she obtains. After Bob receives the classical information, he is immediately certain which operation he needs to perform in order to restore his state to Alice's original state $|\psi\rangle$. Notice that he does not need to have knowledge of the state in order to restore it—he only needs knowledge of the restoration operation.

3. Bob performs the restoration operation: one of the identity, a Pauli $X$ operator, a Pauli $Z$ operator, or the Pauli operator $XZ$, depending on the classical information that he receives from Alice.

Teleportation is an *oblivious* protocol because Alice and Bob do not require any knowledge of the quantum state being teleported in order to perform it. We might also say that this feature of teleportation makes it universal—it works independently of the input state.

You might think that the teleportation protocol violates the no-cloning theorem because a "copy" of the state appears on Bob's system. But this violation does not occur at any point in the protocol because the Bell measurement destroys the information about the state of Alice's original information qubit while recreating it somewhere else. Also, notice that the result of the Bell measurement is independent of the particular probability amplitudes $\alpha$ and $\beta$ corresponding to the state Alice wishes to teleport.

The teleportation protocol is not an instantaneous teleportation, as portrayed in the television episodes of Star Trek. There is no transfer of quantum information instantaneously after the Bell measurement because Bob's local description of the $B$ system is the maximally mixed state $\pi$. It is only after he receives the classical bits to "telecorrect" his state that the transfer occurs. It must be this way—otherwise, they would be able to communicate faster than the speed of light, and superluminal communication is not allowed by the theory of relativity.

Finally, we can phrase the teleportation protocol as a resource inequality:

$$[qq] + 2\,[c \to c] \geq [q \to q]\,. \tag{44}$$

Again, we include only non-local resources in the resource count. The above resource inequality is perhaps the most surprising of the three unit protocols we have studied so far. It combines two resources, noiseless entanglement and noiseless classical communication, that achieve noiseless quantum communication even though they are both individually weaker than it. This protocol and super-dense coding are two of the most fundamental protocols in quantum communication theory because they sparked the notion that there are clever ways of combining resources to generate other resources.

## 4 Three Unit Qudit Protocols

We end this lecture by studying the qudit versions of the three unit protocols. It is useful to have these versions of the protocols because we may want to process qudit systems with them.

The qudit resources are straightforward extensions of the qubit resources. A noiseless qudit channel is the following map:

$$|i\rangle_A \to |i\rangle_B, \tag{45}$$

where $\{|i\rangle_A\}_{i\in\{0,\ldots,d-1\}}$ is some preferred orthonormal basis on Alice's system and $\{|i\rangle_B\}_{i\in\{0,\ldots,d-1\}}$ is some preferred basis on Bob's system. We can also write the qudit channel map as the following isometry:

$$I_{A\to B} \equiv \sum_{i=0}^{d-1} |i\rangle_B \langle i|_A\,. \tag{46}$$

The map $I_{A\to B}$ preserves superposition states so that

$$\sum_{i=0}^{d-1} \alpha_i |i\rangle_A \to \sum_{i=0}^{d-1} \alpha_i |i\rangle_B. \tag{47}$$

A noiseless classical dit channel or *cdit* is the following map:

$$|i\rangle\langle i|_A \to |i\rangle\langle i|_B, \tag{48}$$

$$|i\rangle\langle j|_A \to 0 \text{ for } i \neq j. \tag{49}$$

A noiseless maximally entangled qudit state or an *edit* is as follows:

$$|\Phi\rangle_{AB} \equiv \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_A |i\rangle_B. \tag{50}$$

We quantify the "dit" resources with bit measures. For example, a noiseless qudit channel is the following resource:

$$\log d \, [q \to q] \, , \tag{51}$$

where the logarithm is base two. Thus, one qudit channel can transmit $\log d$ qubits of quantum information so that the qubit remains our standard unit of quantum information. We quantify the amount of information transmitted according to the dimension of the space that is transmitted. For example, suppose that a quantum system has eight levels. We can then encode three qubits of quantum information in this eight-level system.

Likewise, a classical dit channel is the following resource:

$$\log d \, [c \to c] \, , \tag{52}$$

so that a classical dit channel transmits $\log d$ classical bits. The parameter $d$ here is the number of classical messages that the channel transmits.

Finally, an edit is the following resource:

$$\log d \, [qq] \, . \tag{53}$$

We quantify the amount of entanglement in a maximally entangled state by its Schmidt rank. We measure entanglement in units of ebits.

The extension of both the entanglement distribution protocol and the super-dense coding protocol to the qudit case is straightforward, so we will skip them (you can find these discussed in the book).

### 4.1 Quantum Teleportation

The operations in the qudit teleportation protocol are again similar to the qubit case. The protocol proceeds in three steps:

1. Alice possesses an arbitrary qudit $|\psi\rangle_{A'}$ where

$$|\psi\rangle_{A'} \equiv \sum_{i=0}^{d-1} \alpha_i |i\rangle_{A'}. \tag{54}$$

Alice and Bob share a maximally entangled qudit state $|\Phi^+\rangle_{AB}$ of the form

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A |j\rangle_B. \tag{55}$$

The joint state of Alice and Bob is then $|\psi\rangle_{A'} \, |\Phi\rangle_{AB}$. Alice performs a measurement in the basis $\{|\Phi_{i,j}\rangle_{A'A}\}_{i,j}$.

2. She transmits the measurement result $i$, $j$ to Bob with the use of two classical dit channels.

3. Bob then applies the unitary transformation $Z_B(j)X_B(i)$ to his state to "telecorrect" it to Alice's original qudit.

We prove that this protocol works by analyzing the probability of the measurement result and the post-measurement state on Bob's system. The techniques that we employ here are different from those for the qubit case.

First, let us suppose that Alice would like to teleport the $A'$ system of a state $|\psi\rangle_{RA'}$ that she shares with an inaccessible reference system $R$. This way, our teleportation protocol encompasses the most general setting in which Alice would like to teleport a mixed state on $A'$. Also, Alice shares the maximally entangled edit state $|\Phi\rangle_{AB}$ with Bob. Alice first performs a measurement of the systems $A'$ and $A$ in the basis $\{|\Phi_{i,j}\rangle_{A'A}\}_{i,j}$ where

$$|\Phi_{i,j}\rangle_{A'A} = U_{A'}^{ij} |\Phi\rangle_{A'A} , \tag{56}$$

and

$$U_{A'}^{ij} \equiv Z_{A'}(j)X_{A'}(i). \tag{57}$$

The measurement operators are thus

$$|\Phi_{i,j}\rangle\langle\Phi_{i,j}|_{A'A}. \tag{58}$$

Then the unnormalized post-measurement state is

$$|\Phi_{i,j}\rangle\langle\Phi_{i,j}|_{A'A} \; |\psi\rangle_{RA'} |\Phi\rangle_{AB} . \tag{59}$$

We can rewrite this state as follows, by exploiting the definition of $|\Phi_{i,j}\rangle_{A'A}$ in (56):

$$|\Phi_{i,j}\rangle\langle\Phi|_{A'A} \; \left(U_{A'}^{ij}\right)^{\dagger} \; |\psi\rangle_{RA'} |\Phi\rangle_{AB} . \tag{60}$$

Recall the "Bell-state matrix identity" or "transpose trick" that holds for any maximally entangled state $|\Phi\rangle$. We can exploit this result to show that the action of the unitary $\left(U^{ij}\right)^{\dagger}$ on the $A'$ system is the same as the action of the unitary $\left(U^{ij}\right)^{*}$ on the $A$ system:

$$|\Phi_{i,j}\rangle\langle\Phi|_{A'A} \; \left(U_{A}^{ij}\right)^{*} \; |\psi\rangle_{RA'} |\Phi\rangle_{AB} . \tag{61}$$

Then the unitary $\left(U_{A}^{ij}\right)^{*}$ commutes with the systems $R$ and $A'$:

$$|\Phi_{i,j}\rangle\langle\Phi|_{A'A} \; |\psi\rangle_{RA'} \; \left(U_{A}^{ij}\right)^{*} |\Phi\rangle_{AB} . \tag{62}$$

We can again apply the Bell state matrix identity to show that the state is equal to

$$|\Phi_{i,j}\rangle\langle\Phi|_{A'A} \; |\psi\rangle_{RA'} \; \left(U_{B}^{ij}\right)^{\dagger} |\Phi\rangle_{AB} . \tag{63}$$

Then we can commute the unitary $\left(U_{B}^{ij}\right)^{\dagger}$ all the way to the left, and we can switch the order of $|\psi\rangle_{RA'}$ and $|\Phi\rangle_{AB}$ without any problem because the system labels are sufficient to track the states in these systems:

$$\left(U_{B}^{ij}\right)^{\dagger} |\Phi_{i,j}\rangle\langle\Phi|_{A'A} \; |\Phi\rangle_{AB} \; |\psi\rangle_{RA'}. \tag{64}$$

Now let us consider the very special overlap $\langle\Phi|_{A'A}\ |\Phi\rangle_{AB}$ of the maximally entangled edit state with itself on different systems:

$$\langle\Phi|_{A'A}\ |\Phi\rangle_{AB} = \left(\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\langle i|_{A'}\langle i|_A\right)\left(\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle_A|j\rangle_B\right) \tag{65}$$

$$= \frac{1}{d}\sum_{i,j=0}^{d-1}\langle i|_{A'}\langle i|_A|j\rangle_A\,|j\rangle_B \tag{66}$$

$$= \frac{1}{d}\sum_{i,j=0}^{d-1}\langle i|_{A'}\,\langle i|j\rangle_A\,|j\rangle_B \tag{67}$$

$$= \frac{1}{d}\sum_{i=0}^{d-1}\langle i|_{A'}|i\rangle_B \tag{68}$$

$$= \frac{1}{d}\sum_{i=0}^{d-1}|i\rangle_B\langle i|_{A'} \tag{69}$$

$$= \frac{1}{d}I_{A'\to B}. \tag{70}$$

The first equality follows by definition. The second equality follows from linearity and rearranging terms in the multiplication and summation. The third and fourth equalities follow by realizing that $\langle i|_A|j\rangle_A$ is an inner product and evaluating it for the orthonormal basis $\{|i\rangle_A\}$. The fifth equality follows by rearranging the bra and the ket. The final equality is our last important realization: the operator $\sum_{i=0}^{d-1}|i\rangle_B\langle i|_{A'}$ is the noiseless qudit channel $I_{A'\to B}$ that the teleportation protocol creates from the system $A'$ to $B$ (see the definition of a noiseless qudit channel in (46))). We might refer to this as the "teleportation map."

We now apply the teleportation map to the state in (64):

$$\left(U_B^{ij}\right)^\dagger|\Phi_{i,j}\rangle\langle\Phi|_{A'A}\ |\Phi\rangle_{AB}\ |\psi\rangle_{RA'} = \left(U_B^{ij}\right)^\dagger|\Phi_{i,j}\rangle_{A'A}\frac{1}{d}I_{A'\to B}\ |\psi\rangle_{RA'} \tag{71}$$

$$= \frac{1}{d}\left(U_B^{ij}\right)^\dagger|\Phi_{i,j}\rangle_{A'A}|\psi\rangle_{RB} \tag{72}$$

$$= \frac{1}{d}|\Phi_{i,j}\rangle_{A'A}\ \left(U_B^{ij}\right)^\dagger|\psi\rangle_{RB}. \tag{73}$$

We can compute the probability of receiving outcome $i$ and $j$ from the measurement when the input state is $|\psi\rangle_{RA'}$. It is just equal to the overlap of the above vector with itself:

$$p\left(i,j|\psi\right) = \left[\frac{1}{d}\langle\Phi_{i,j}|_{A'A}\ \langle\psi|_{RB}\,U_B^{ij}\right]\left[\frac{1}{d}|\Phi_{i,j}\rangle_{A'A}\ \left(U_B^{ij}\right)^\dagger|\psi\rangle_{RB}\right] \tag{74}$$

$$= \frac{1}{d^2}\langle\Phi_{i,j}|_{A'A}|\Phi_{i,j}\rangle_{A'A}\ \langle\psi|_{RB}\,U_B^{ij}\left(U_B^{ij}\right)^\dagger|\psi\rangle_{RB} \tag{75}$$

$$= \frac{1}{d^2}\langle\Phi_{i,j}|_{A'A}|\Phi_{i,j}\rangle_{A'A}\ \langle\psi|_{RB}\,|\psi\rangle_{RB} \tag{76}$$

$$= \frac{1}{d^2}. \tag{77}$$

Thus, the probability of the outcome $(i,j)$ is completely random and independent of the input state. We would expect this to be the case for a universal teleportation protocol that operates independently of the input state. Thus, after normalization, the state on Alice and Bob's system is

$$|\Phi_{i,j}\rangle_{A'A} \left(U_B^{ij}\right)^{\dagger} |\psi\rangle_{RB} . \tag{78}$$

At this point, Bob does not know the result of the measurement. We obtain his density operator by tracing over the systems $A'$, $A$, and $R$ to which he does not have access and taking the expectation over all the measurement outcomes:

$$\begin{aligned}
&\mathrm{Tr}_{A'AR}\left\{\frac{1}{d^2}\sum_{i,j=0}^{d-1} |\Phi_{i,j}\rangle\langle\Phi_{i,j}|_{A'A} \left(U_B^{ij}\right)^{\dagger} |\psi\rangle\langle\psi|_{RB} U_B^{ij}\right\} \\
&= \frac{1}{d^2}\sum_{i,j=0}^{d-1} \left(U_B^{ij}\right)^{\dagger} \psi_B U_B^{ij} &(79)\\
&= \pi_B. &(80)
\end{aligned}$$

The first equality follows by evaluating the partial trace and by defining

$$\psi_B \equiv \mathrm{Tr}_R\left\{|\psi\rangle\langle\psi|_{RB}\right\}. \tag{81}$$

The second equality follows because applying a Heisenberg–Weyl operator uniformly at random completely randomizes a quantum state to be the maximally mixed state.

Now suppose that Alice sends the measurement results $i$ and $j$ over two uses of a noiseless classical dit channel. Bob then knows that the state is

$$\left(U_B^{ij}\right)^{\dagger} |\psi\rangle_{RB}, \tag{82}$$

and he can apply $U_B^{ij}$ to make the overall state become $|\psi\rangle_{RB}$. This final step completes the teleportation process. The resource inequality for the qudit teleportation protocol is as follows:

$$\log d\,[qq] + 2\log d\,[c \to c] \geq \log d\,[q \to q]\,. \tag{83}$$