PHYS 7895: Quantum Information Theory	Fall 2015

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1 Overview

In the last lecture we motivated the model of quantum channels that we will take for the rest of the course: they are linear, completely positive, trace-preserving maps. These three constraints imply a structure for any quantum channel—that it has a representation in terms of Kraus operators (this result is known as the Choi-Kraus theorem).

In this lecture, we show how everything we have considered so far can be viewed as a quantum channel, including preparation of states and measurements of states. In this sense, quantum channels are the most general objects that we deal with in quantum information theory. We also discuss how to combine channels and the notion of the adjoint of a quantum channel.

2 Combining Channels

2.1 Serial Concatenation of Quantum Channels

A quantum state may undergo not just one type of quantum evolution—it can of course undergo one quantum channel followed by another quantum channel. Let $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ denote a first quantum channel and let $\mathcal{M} : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_C)$ denote a second quantum channel. Suppose that the Kraus operators of $\mathcal{N}$ are $\{N_k\}$ and the Kraus operators of $\mathcal{M}$ are $\{M_k\}$. It is straightforward to define the serial concatenation $\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B}$ of these two quantum channels. Consider that the output of the first channel is

$$\mathcal{N}_{A\to B}(\rho_A) \equiv \sum_k N_k \rho_A N_k^\dagger, \tag{1}$$

for some input density operator $\rho_A \in \mathcal{D}(\mathcal{H}_A)$. The output of the serially concatenated channel $\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B}$ is then

$$\left(\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B}\right)(\rho_A) = \sum_k M_k \mathcal{N}_{A\to B}(\rho) M_k^\dagger = \sum_{k,k'} M_k N_{k'} \rho_A N_{k'}^\dagger M_k^\dagger. \tag{2}$$

It is clear that the Kraus operators of the serially concatenated channel $\mathcal{M}_{B\to C} \circ \mathcal{N}_{A\to B}$ are $\{M_k N_{k'}\}_{k,k'}$. Serial concatenation of channels has an obvious generalization to a serial concatenation of more than two channels.

2.2 Parallel Concatenation of Quantum Channels

We can also use two channels in parallel. That is, suppose that we send a system A through a channel $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_C)$ and a system B through a channel $\mathcal{M} : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_D)$. Suppose further that the Kraus operators of $\mathcal{N}_{A\to C}$ are $\{N_k\}$ and those for $\mathcal{M}_{B\to D}$ are $\{M_{k'}\}$. Then the parallel concatenation of the two channels has the following action on an input density operator $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$:

$$(\mathcal{N}_{A\to C} \otimes \mathcal{M}_{B\to D})(\rho_{AB}) = \sum_{k,k'} (N_k \otimes M_{k'}) (\rho_{AB}) (N_k \otimes M_{k'})^\dagger , \tag{3}$$

so that the Kraus operators of the parallel concatenated channel $\mathcal{N} \otimes \mathcal{M}$ are $\{N_k \otimes M_{k'}\}$. Parallel concatenation of channels also has an obvious generalization to more than two channels.

2.3 Unital Maps and Adjoints of Quantum Channels

Recall that the adjoint $G^\dagger$ of a linear operator G is defined as the unique linear operator satisfying the following set of equations:

$$\langle y, Gx\rangle = \left\langle G^\dagger y, x \right\rangle , \tag{4}$$

for all vectors x and y, and with $\langle z, w\rangle = \sum_i z_i^* w_i$ defined as the inner product between vectors z and w.

As an extension of this idea, we can define an inner product for operators:

Definition 1 (Hilbert–Schmidt inner product)**.** *The Hilbert–Schmidt inner product between two operators $C, D \in \mathcal{B}(\mathcal{H})$ is defined as follows:*

$$\langle C, D\rangle \equiv \mathrm{Tr}\{C^\dagger D\}. \tag{5}$$

This then allows us to define the adjoint $\mathcal{N}^\dagger$ of a linear map $\mathcal{N}$ in a way similar to (4):

Definition 2 (Adjoint Map)**.** *Let $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a linear map. The adjoint $\mathcal{N}^\dagger : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ of a linear map $\mathcal{N}$ is the unique linear map satisfying the following set of equations:*

$$\langle Y, \mathcal{N}(X)\rangle = \left\langle \mathcal{N}^\dagger(Y), X \right\rangle , \tag{6}$$

for all $X \in \mathcal{B}(\mathcal{H}_A)$ and $Y \in \mathcal{B}(\mathcal{H}_B)$.

Another important class of linear maps are unital maps, defined as follows:

Definition 3 (Unital Map)**.** *A linear map $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is unital if it preserves the identity operator, in the sense that $\mathcal{N}(I_A) = I_B$.*

Given the notion of an adjoint map, it is natural to inquire what is the adjoint of a quantum channel, and furthermore, what is an interpretation of it. So let us now suppose that $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is a quantum channel with a set $\{V_l\}$ of Kraus operators satisfying $\sum_l V_l^\dagger V_l = I_A$. Then we compute

$$\langle Y, \mathcal{N}(X)\rangle = \operatorname{Tr}\left\{Y^\dagger \sum_l V_l X V_l^\dagger\right\} = \operatorname{Tr}\left\{\sum_l V_l^\dagger Y^\dagger V_l X\right\} \tag{7}$$

$$= \operatorname{Tr}\left\{\left(\sum_l V_l^\dagger Y V_l\right)^\dagger X\right\} = \left\langle \sum_l V_l^\dagger Y V_l, X\right\rangle, \tag{8}$$

where the second equality is from linearity and cyclicity of trace and the last is from the definition of the Hilbert–Schmidt inner product. Thus, the adjoint $\mathcal{N}^\dagger$ of any quantum channel $\mathcal{N}$ is given by

$$\mathcal{N}^\dagger(Y) = \sum_l V_l^\dagger Y V_l. \tag{9}$$

The adjoint $\mathcal{N}^\dagger$ is completely positive, as one can verify. Furthermore, the adjoint $\mathcal{N}^\dagger$ is unital because

$$\mathcal{N}^\dagger(I_B) = \sum_l V_l^\dagger I_B V_l = \sum_l V_l^\dagger V_l = I_A. \tag{10}$$

We summarize these results as follows:

Proposition 4. *The adjoint $\mathcal{N}^\dagger : \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ of a quantum channel $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is a completely positive, unital map.*

What is an interpretation of the adjoint of a quantum channel? It provides a connection from the Schrödinger picture of quantum physics, in which the focus is on the evolution of states, to the Heisenberg picture, in which the focus is on the evolution of observables or measurement operators. To see this, let $\{\Lambda_B^j\}$ be a POVM, ρ_A be a density operator, and $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a quantum channel. Suppose that we prepare the state ρ_A, apply the channel $\mathcal{N}$, and then perform the measurement $\{\Lambda_B^j\}$. The probability of getting outcome j from the measurement is given by the Born rule:

$$p_J(j) = \operatorname{Tr}\{\Lambda_B^j \mathcal{N}(\rho_A)\} = \operatorname{Tr}\{\mathcal{N}^\dagger(\Lambda_B^j)\rho_A\}, \tag{11}$$

where the second equality follows because $\mathcal{N}^\dagger$ is the adjoint of $\mathcal{N}$. This latter expression is what corresponds to the Heisenberg picture. Here, the interpretation is that each measurement operator Λ_B^j "evolves backwards" to become $\mathcal{N}^\dagger(\Lambda_B^j)$ and then the measurement $\{\mathcal{N}^\dagger(\Lambda_B^j)\}$ is performed on the state ρ_A. We should verify that the set $\{\mathcal{N}^\dagger(\Lambda_B^j)\}$ indeed constitutes a measurement. Consider that each $\mathcal{N}^\dagger(\Lambda_B^j)$ is positive semi-definite, given that the adjoint is a completely positive map, and that

$$\sum_j \mathcal{N}^\dagger(\Lambda_B^j) = \mathcal{N}^\dagger\left(\sum_j \Lambda_B^j\right) = \mathcal{N}^\dagger(I_B) = I_A, \tag{12}$$

where the equalities are following because $\mathcal{N}^\dagger$ is linear and unital. The interpretation of the measurement $\{\mathcal{N}^\dagger(\Lambda_B^j)\}$ is that it is the physical procedure corresponding to applying the channel $\mathcal{N}$ and then performing the measurement $\{\Lambda_B^j\}$, which is of course a valid measurement procedure.

3 Interpretations of Quantum Channels

We now detail two interpretations of quantum channels that are consistent with the Choi–Kraus theorem. The first is that we can intepret the noise occurring in a quantum channel as the loss of a measurement outcome, and the second is that we can interpret noise as being due to a unitary interaction with an environment to which we do not have access.

3.1 Noisy Evolution as the Loss of a Measurement Outcome

We can interpret the noise a quantum channel as arising from the loss of a measurement outcome (see Figure 1). Suppose that we have an ensemble of states $\{p_X(x), |\psi_x\rangle\}_{x\in\mathcal{X}}$ and we perform a measurement with a set $\{M_k\}$ of measurement operators for which $\sum_k M_k^\dagger M_k = I$. First let us suppose that we know that the state is $|\psi_x\rangle$. Then the probability of obtaining the measurement outcome k is $p_{K|X}(k|x)$ where

$$p_{K|X}(k|x) = \langle\psi_x|M_k^\dagger M_k|\psi_x\rangle, \tag{13}$$

and the post-measurement state is

$$\frac{M_k|\psi_x\rangle}{\sqrt{p_{K|X}(k|x)}}. \tag{14}$$

Let us now suppose that we lose track of the measurement outcome, or equivalently, someone else measures the system and does not inform us of the measurement outcome. The resulting ensemble description is then

$$\left\{p_{X|K}(x|k)p_K(k), M_k|\psi_x\rangle/\sqrt{p_{K|X}(k|x)}\right\}_{x\in\mathcal{X},k}. \tag{15}$$

The density operator of the ensemble is then

$$\begin{aligned}
&\sum_{x,k} p_{X|K}(x|k)p_K(k)\frac{M_k|\psi_x\rangle\langle\psi_x|M_k^\dagger}{p_{K|X}(k|x)} \\
&= \sum_{x,k} p_{K|X}(k|x)p_X(x)\frac{M_k|\psi_x\rangle\langle\psi_x|M_k^\dagger}{p_{K|X}(k|x)} && (16)\\
&= \sum_{x,k} p_X(x)M_k|\psi_x\rangle\langle\psi_x|M_k^\dagger && (17)\\
&= \sum_k M_k\rho M_k^\dagger. && (18)
\end{aligned}$$

We can thus write this evolution as a quantum channel $\mathcal{N}(\rho)$ where

$$\mathcal{N}(\rho) = \sum_k M_k\rho M_k^\dagger. \tag{19}$$

The measurement operators are playing the role of Kraus operators in this evolution.

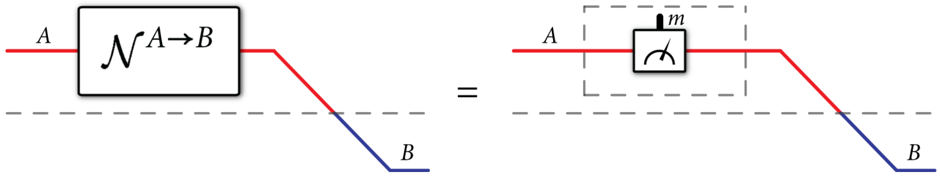

Figure 1: The diagram on the left depicts a quantum channel $\mathcal{N}_{A\rightarrow B}$ that takes a quantum system A to a quantum system B. This quantum channel has an interpretation in terms of the diagram on the right, in which some third party performs a measurement on the input system and does not inform the receiver of the measurement outcome.

3.2 Noisy Evolution from a Unitary Interaction

There is another perspective on quantum noise that is helpful to consider. Suppose that a quantum system A begins in the state ρ_A and that there is an environment system E in a pure state $|0\rangle_E$. So the initial state of the joint system AE is $\rho_A \otimes |0\rangle\langle 0|_E$. Suppose that these two systems interact according to some unitary operator U_{AE} acting on both systems A and E. If we only have access to the system A after the interaction, then we calculate the state σ_A of this system by taking the partial trace over the environment E:

$$\sigma_A = \text{Tr}_E \left\{ U_{AE} \left(\rho_A \otimes |0\rangle\langle 0|_E \right) U_{AE}^\dagger \right\}. \tag{20}$$

This evolution is equivalent to that of a completely positive, trace-preserving map with Kraus operators

$$\{ B_i \equiv \left(I_A \otimes \langle i|_E \right) U_{AE} \left(I_A \otimes |0\rangle_E \right) \}_i. \tag{21}$$

This follows easily because we can take the partial trace with respect to an orthonormal basis $\{|i\rangle_E\}$ for the environment:

$$\begin{aligned}
&\text{Tr}_E \left\{ U_{AE} \left(\rho_A \otimes |0\rangle \langle 0|_E \right) U_{AE}^\dagger \right\} \\
&= \sum_i \left(I_A \otimes \langle i|_E \right) U_{AE} \left(\rho_A \otimes |0\rangle\langle 0|_E \right) U_{AE}^\dagger \left(I_A \otimes |i\rangle_E \right) && (22) \\
&= \sum_i \left(I_A \otimes \langle i|_E \right) U_{AE} \left(I_A \otimes |0\rangle_E \right) \left(\rho_A \right) \left(I_A \otimes \langle 0|_E \right) U_{AE}^\dagger \left(I_A \otimes |i\rangle_E \right) && (23) \\
&= \sum_i B_i \rho_A B_i^\dagger. && (24)
\end{aligned}$$

The first equality follows from the definition for partial trace. The second equality follows because

$$\rho_A \otimes |0\rangle\langle 0|_E = \left(I_A \otimes |0\rangle_E \right) \left(\rho_A \right) \left(I_A \otimes \langle 0|_E \right). \tag{25}$$

That the operators $\{B_i\}$ are a legitimate set of Kraus operators satisfying $\sum_i B_i^\dagger B_i = I_A$ follows from the unitarity of U_{AE} and the orthonormality of the basis $\{|i\rangle_E\}$:

$$\begin{aligned}
&\sum_i B_i^\dagger B_i \\
&= \sum_i \left(I_A \otimes \langle 0|_E\right) U_{AE}^\dagger \left(I_A \otimes |i\rangle_E\right) \left(I_A \otimes \langle i|_E\right) U_{AE} \left(I_A \otimes |0\rangle_E\right) && (26)\\
&= \left(I_A \otimes \langle 0|_E\right) U_{AE}^\dagger \left(I_A \otimes \sum_i |i\rangle\langle i|_E\right) U_{AE} \left(I_A \otimes |0\rangle_E\right) && (27)\\
&= \left(I_A \otimes \langle 0|_E\right) U_{AE}^\dagger U_{AE} \left(I_A \otimes |0\rangle_E\right) && (28)\\
&= \left(I_A \otimes \langle 0|_E\right) I_A \otimes I_E \left(I_A \otimes |0\rangle_E\right) && (29)\\
&= I_A. && (30)
\end{aligned}$$

4 Quantum Channels are All-Encompassing

In this section, we show how everything we have considered so far can be viewed as a quantum channel. This includes physical evolutions as we have discussed so far, but additionally (and perhaps surprisingly) density operators, discarding of systems, and quantum measurements. From this perspective, one could argue that that there really is just a single underlying postulate of quantum physics, that everything we consider in the theory is just a quantum channel of some sort.

4.1 Preparation and Appending Channels

The preparation of a system A in a state $\rho_A \in \mathcal{D}(\mathcal{H}_A)$ is a particular type of quantum channel, with trivial input Hilbert space $\mathbb{C}$ and output Hilbert space $\mathcal{H}_A$. Let $\rho_A = \sum_x p_X(x)|x\rangle\langle x|_A$ be a spectral decomposition of ρ_A. Then the Kraus operators of this channel are $\{N_x \equiv \sqrt{p_X(x)}|x\rangle_A\}$, and we can easily verify that these are legitimate Kraus operators by calculating

$$\sum_x N_x^\dagger N_x = \sum_x \left(\sqrt{p_X(x)}\langle x|_A\right)\left(\sqrt{p_X(x)}|x\rangle_A\right) = \sum_x p_X(x) = 1, \tag{31}$$

so that the completeness relation holds, given that the number 1 is the identity for the trivial Hilbert space $\mathbb{C}$. Considering that the number 1 is also the only density operator in $\mathcal{D}(\mathbb{C})$, we can view this channel as mapping the trivial density operator 1 to a density operator $\rho_A \in \mathcal{D}(\mathcal{H}_A)$. It is thus a preparation channel.

Definition 5 (Preparation Channel)**.** *A preparation channel $\mathcal{P}_A \equiv \mathcal{P}_{\mathbb{C}\to A}$ prepares a quantum system A in a given state $\rho_A \in \mathcal{D}(\mathcal{H}_A)$.*

This leads to a related channel, called an appending channel:

Definition 6 (Appending Channel)**.** *An appending channel is the parallel concatenation of the identity channel and a preparation channel.*

Thus, an appending channel has the following action on a system B in the state σ_B:

$$\left(\mathcal{P}_A \otimes \mathrm{id}_B\right)(\sigma_B) = \rho_A \otimes \sigma_B. \tag{32}$$

The Kraus operators of such an appending channel are then $\{\sqrt{p_X(x)}|x\rangle_A \otimes I_B\}$.

4.2 Trace-out and Discarding Channels

In some sense, the opposite of preparation is discarding. So suppose that we completely discard the contents of a quantum system A. The channel that does so is called a *trace-out channel* Tr_A, and its action is to map any density operator $\rho_A \in \mathcal{D}(\mathcal{H}_A)$ to the trivial density operator 1. The Kraus operators of the trace-out channel are $\{N_x \equiv \langle x|_A\}$, where $\{|x\rangle_A\}$ is some orthonormal basis for the system A. These Kraus operators satisfy the completeness relation because

$$\sum_x N_x^\dagger N_x = \sum_x |x\rangle\langle x|_A = I_A. \tag{33}$$

This channel is in direct correspondence with the trace operation.

Now suppose that we have two systems A and B, and we would like to discard system A only. The channel that does is a *discarding channel*, which is the parallel concatenation of the trace-out channel Tr_A and the identity channel id_B. It has the following action on a density operator $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$:

$$(\mathrm{Tr}_A \otimes \mathrm{id}_B)(\rho_{AB}) = \sum_x (\langle x|_A \otimes I_B)\, \rho_{AB}\, (|x\rangle_A \otimes I_B) = \mathrm{Tr}_A\{\rho_{AB}\}, \tag{34}$$

where we have taken the Kraus operators of $\mathrm{Tr}_A \otimes \mathrm{id}_B$ to be $\{\langle x|_A \otimes I_B\}$. Clearly, this channel is in direct correspondence with the partial trace operation.

4.3 Unitary Channels

Unitary evolution is a special kind of quantum channel in which there is a single Kraus operator U, satisfying $UU^\dagger = U^\dagger U = I$. Unitary channels are thus completely positive, trace-preserving, and unital. Let $\rho \in \mathcal{D}(\mathcal{H})$. Under the action of a unitary channel $\mathcal{U}$, this state evolves as

$$\mathcal{U}(\rho) = U\rho U^\dagger. \tag{35}$$

Our convention henceforth is to denote a unitary channel by $\mathcal{U}$ and a unitary operator by U.

4.4 Classical-to-Classical Channels

It is natural to expect that classical channels are special cases of quantum channels, and indeed, this is the case. To see this, fix an input probability distribution $p_X(x)$ and a classical channel $p_{Y|X}(y|x)$. Fix an orthonormal basis $\{|x\rangle\}$ corresponding to the input letters and an orthonormal basis $\{|y\rangle\}$ corresponding to the output letters. We can then encode the input probability distribution $p_X(x)$ as a density operator ρ of the following form:

$$\rho = \sum_x p_X(x)|x\rangle\langle x|. \tag{36}$$

Let $\mathcal{N}$ be a quantum channel with the following Kraus operators

$$\left\{\sqrt{p_{Y|X}(y|x)}|y\rangle\langle x|\right\}_{x,y}. \tag{37}$$

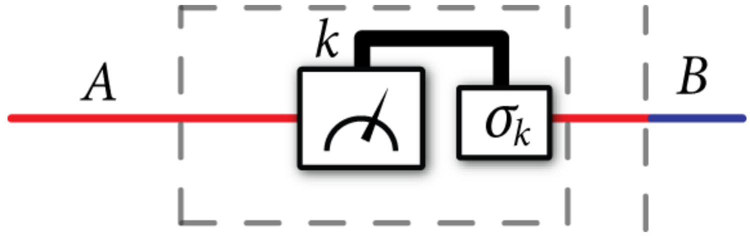

Figure 2: This figure illustrates the internal workings of a classical-quantum channel. It first measures the input state in some basis $\{|k\rangle\}$ and outputs a quantum state σ_k conditional on the measurement outcome.

(The fact that these are legitimate Kraus operators follows directly from the fact that $p_{Y|X}(y|x)$ is a conditional probability distribution.) The quantum channel then has the following action on the input ρ:

$$\mathcal{N}(\rho) = \sum_{x,y} \sqrt{p_{Y|X}(y|x)}|y\rangle\langle x| \left(\sum_{x'} p_X(x') \left|x'\right\rangle \left\langle x'\right| \right) \sqrt{p_{Y|X}(y|x)}|x\rangle\langle y| \tag{38}$$

$$= \sum_{x,y,x'} p_{Y|X}(y|x) p_X(x') \left|\langle x'|x\rangle\right|^2 \ |y\rangle\langle y| \tag{39}$$

$$= \sum_{x,y} p_{Y|X}(y|x) p_X(x) \ |y\rangle\langle y| \tag{40}$$

$$= \sum_{y} \left(\sum_{x} p_{Y|X}(y|x) p_X(x) \right) \ |y\rangle\langle y|. \tag{41}$$

Thus, the evolution is the same that a noisy classical channel $p_{Y|X}(y|x)$ would enact on a probability distribution $p_X(x)$ by taking it to

$$p_Y(y) = \sum_x p_{Y|X}(y|x) p_X(x) \tag{42}$$

at the output.

Since a noiseless classical channel has $p_{Y|X}(y|x) = \delta_{x,y}$, we are led to the following definition:

Definition 7 (Noiseless Classical Channel)**.** *Let $\{|x\rangle\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. A noiseless classical channel has the following action on a density operator $\rho \in \mathcal{D}(\mathcal{H})$:*

$$\rho \to \sum_x |x\rangle\langle x|\rho|x\rangle\langle x|. \tag{43}$$

That is, it removes the off-diagonal elements of ρ when represented as a matrix with respect to the basis $\{|x\rangle\}$.

4.5 Classical-to-Quantum Channels

Classical-to-quantum channels, or classical–quantum channels for short, are channels which take classical systems to quantum systems. They thus go one step beyond both classical-to-classical channels and preparation channels. More generally, they make a given quantum system classical and then prepare a quantum state, as discussed in the following definition:

Definition 8 (Classical–Quantum Channel)**.** *A classical–quantum channel first measures the input state in a particular orthonormal basis and outputs a density operator conditioned on the result of the measurement. Given an orthonormal basis* $\{|k\rangle_A\}$ *and a set of states* $\{\sigma_B^k\}$*, each of which is in* $\mathcal{D}(\mathcal{H}_B)$*, a classical–quantum channel has the following action on an input density operator* $\rho_A \in \mathcal{D}(\mathcal{H}_A)$*:*

$$\rho_A \to \sum_k \langle k|_A \rho_A |k\rangle_A \sigma_B^k. \tag{44}$$

Let us see how this comes about, using the definition above. The classical–quantum channel first measures the input state ρ_A in the basis $\{|k\rangle_A\}$. Given that the result of the measurement is k, the post measurement state is

$$\frac{|k\rangle\langle k|\rho_A|k\rangle\langle k|}{\langle k|\rho_A|k\rangle}. \tag{45}$$

The channel then correlates a density operator σ_B^k with the post-measurement state k:

$$\frac{|k\rangle\langle k|\rho_A|k\rangle\langle k|}{\langle k|\rho_A|k\rangle} \otimes \sigma_B^k. \tag{46}$$

This action leads to an ensemble:

$$\left\{\langle k|\rho_A|k\rangle, \frac{|k\rangle\langle k|\rho_A|k\rangle\langle k|}{\langle k|\rho_A|k\rangle} \otimes \sigma_B^k\right\}, \tag{47}$$

and the density operator of the ensemble is

$$\sum_k \langle k|\rho_A|k\rangle \frac{|k\rangle\,\langle k|\rho_A|k\rangle\langle k|}{\langle k|\rho_A|k\rangle} \otimes \sigma_B^k = \sum_k |k\rangle\langle k|\rho_A|k\rangle\langle k| \otimes \sigma_B^k. \tag{48}$$

The channel then only outputs the system on the right (tracing out the first system) so that the resulting channel is as given in (44).

Exercise 9. *What are a set of Kraus operators for a classical–quantum channel?*

4.6 Quantum-to-Classical Channels (Measurement Channels)

Quantum-to-classical, or quantum–classical channels for short, are in some sense the opposite of classical–quantum channels. They take a quantum system to a classical one, and as such, they are in direct correspondence with measurements. So sometimes they are referred to as measurement channels. They also represent a way of generalizing classical channels different from classical–quantum channels.

Definition 10 (Quantum–Classical Channels)**.** *Let* $\{|x\rangle_X\}$ *be an orthonormal basis for a Hilbert space* $\mathcal{H}_X$*, and let* $\{\Lambda_A^x\}$ *be a POVM acting on the system A. A quantum–classical channel has the following action on an input density operator* $\rho_A \in \mathcal{D}(\mathcal{H}_A)$*:*

$$\rho_A \to \sum_x \text{Tr}\{\Lambda_A^x \rho_A\} |x\rangle\langle x|_X. \tag{49}$$

We should verify that this is indeed a quantum channel, by determining its Kraus operators. Consider that the trace operation can be written as $\text{Tr}\{\cdot\} = \sum_j \langle j|_A \cdot |j\rangle_A$, where $\{|j\rangle_A\}$ is some orthonormal basis for $\mathcal{H}_A$. Then we can rewrite (49) as

$$\sum_x \text{Tr}\{\Lambda_A^x \rho_A\}|x\rangle\langle x|_X = \sum_x \text{Tr}\left\{\sqrt{\Lambda_A^x}\rho_A\sqrt{\Lambda_A^x}\right\}|x\rangle\langle x|_X \tag{50}$$

$$= \sum_{x,j} \langle j|_A \sqrt{\Lambda_A^x}\rho_A\sqrt{\Lambda_A^x}|j\rangle_A |x\rangle\langle x|_X \tag{51}$$

$$= \sum_{x,j} |x\rangle_X \langle j|_A \sqrt{\Lambda_A^x}\rho_A\sqrt{\Lambda_A^x}|j\rangle_A \langle x|_X. \tag{52}$$

So this development reveals that a set of Kraus operators for the channel in (49) are $\{N_{x,j} \equiv |x\rangle_X\langle j|_A\sqrt{\Lambda_A^x}\}$. Let us verify the completeness relation for them:

$$\sum_{x,j} N_{x,j}^\dagger N_{x,j} = \sum_{x,j} \sqrt{\Lambda_A^x}|j\rangle_A\langle x|_X|x\rangle_X\langle j|_A\sqrt{\Lambda_A^x} = \sum_{x,j} \sqrt{\Lambda_A^x}|j\rangle_A\langle j|_A\sqrt{\Lambda_A^x} \tag{53}$$

$$= \sum_x \Lambda_A^x = I_A, \tag{54}$$

where the last equality follows because $\{\Lambda_A^x\}$ is a POVM.