PHYS 7895: Quantum Information Theory — Fall 2015

Lecture 9 — September 23, 2015

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1 Overview

In the last lecture we discussed more general models of measurements, the POVM formalism, product states, separable states, and the partial trace operation.

The evolution of a quantum state is never perfect. In this lecture, we discuss the most general approach to understanding quantum evolutions: the *axiomatic approach*. In this powerful approach, we start with three physically reasonable axioms that should hold for any quantum evolution and then deduce a set of mathematical constraints that any quantum evolution should satisfy (this is known as the *Choi-Kraus theorem*). For the rest of the course, we will refer to quantum evolutions satisfying these constraints as *quantum channels*.

2 Axiomatic Approach to Noisy Quantum Evolutions

We now discuss a powerful approach to understanding quantum physical evolutions called the *axiomatic approach*. Here we make three physically reasonable assumptions that any quantum evolution should satisfy and then prove that these axioms imply mathematical constraints on the form of any quantum physical evolution.

All of the constraints we impose are motivated by the reasonable requirement for the output of the evolution to be a quantum state (density operator) if the input to the evolution is a quantum state (density operator). An important assumption to clarify at the outset is that we are viewing a quantum physical evolution as a "black box," meaning that Alice can prepare any state that she wishes before the evolution begins, including pure states or mixed states. Critically, we even allow her to input one share of an entangled state. This is a standard assumption in quantum information theory, but one could certainly question whether this assumption is reasonable. If we do accept this criterion as physically reasonable, then the Choi-Kraus representation theorem for quantum evolutions follows as a consequence.

Notation 1 (Density Operators and Linear Operators)**.** *Let $\mathcal{D}(\mathcal{H})$ denote the space of density operators acting on a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the space of square linear operators acting on $\mathcal{H}$, and let $\mathcal{B}(\mathcal{H}_A, \mathcal{H}_B)$ denote the space of linear operators taking a Hilbert space $\mathcal{H}_A$ to a Hilbert space $\mathcal{H}_B$.*

Throughout this development, we let $\mathcal{N}$ denote a map which takes density operators in $\mathcal{D}(\mathcal{H}_A)$ to those in $\mathcal{D}(\mathcal{H}_B)$. In general, the respective input and output Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ need not be the same. Implicitly, we have already stated a first physically reasonable requirement that we impose on $\mathcal{N}$, namely, that $\mathcal{N}(\rho_A) \in \mathcal{D}(\mathcal{H}_B)$ if $\rho_A \in \mathcal{D}(\mathcal{H}_A)$. Extending this requirement, we demand that $\mathcal{N}$ should be *convex linear* when acting on $\mathcal{D}(\mathcal{H}_A)$:

$$\mathcal{N}(\lambda\rho_A + (1-\lambda)\sigma_A) = \lambda\mathcal{N}(\rho_A) + (1-\lambda)\mathcal{N}(\sigma_A), \tag{1}$$

where $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$ and $\lambda \in [0,1]$.

The physical interpretation of this convex-linearity requirement is in terms of repeated experiments. Suppose a large number of experiments are conducted in which identical quantum systems are prepared in the state ρ_A for a fraction λ of the experiments and in the state σ_A for the other fraction $1-\lambda$ of the experiments. Suppose further that it is not revealed which states are prepared for which experiments. Before you are allowed to perform measurements on each system, the evolution $\mathcal{N}$ is applied to each of the systems. The density operator characterizing the state of each system for these experiments is then $\mathcal{N}(\lambda\rho_A + (1-\lambda)\sigma_A)$. You are then allowed to perform a measurement on each system, which after a large number of experiments allow you to infer that the density operator is $\mathcal{N}(\lambda\rho_A + (1-\lambda)\sigma_A)$. Now, in principle, it could have been revealed which fraction of the experiments had the state ρ_A prepared and which fraction had σ_A prepared. In this case, the density operator describing the ρ_A experiments would be $\mathcal{N}(\rho_A)$ and that describing the σ_A experiments would be $\mathcal{N}(\sigma_A)$. So, it is reasonable to expect that the statistics observed in your measurement outcomes in the first scenario would be consistent with those observed in the second scenario, and this is the physical statement that the requirement (1) makes.

Now, it is mathematically convenient to extend the domain and range of the quantum channel to apply not only to density operators but to all linear operators. To this end, it is possible to find a unique linear extension $\widetilde{\mathcal{N}}$ of any quantum evolution $\mathcal{N}$ defined as above (originally defined exclusively by its action on density operators and satisfying convex linearity). See Section 3 for a full development of this idea. Thus, it is reasonable to associate this unique linear extension $\widetilde{\mathcal{N}}$ to the quantum physical evolution $\mathcal{N}$ mathematically, and in what follows (and for the rest of the book), we simply identify a physical evolution $\mathcal{N}$ with its unique linear extension $\widetilde{\mathcal{N}}$, and this is what we call a *quantum channel*. For these reasons, we now impose that any quantum channel $\mathcal{N}$ is linear:

Criterion 2 (Linearity)**.** *A quantum channel $\mathcal{N}$ is a linear map:*

$$\mathcal{N}(\alpha X_A + \beta Y_A) = \alpha\mathcal{N}(X_A) + \beta\mathcal{N}(Y_A), \tag{2}$$

where $X_A, Y_A \in \mathcal{B}(\mathcal{H}_A)$ and $\alpha, \beta \in \mathbb{C}$.

We have already demanded that quantum physical evolutions should take density operators to density operators. Combining with linearity (in particular, scale invariance) implies that quantum channels should preserve the class of positive semi-definite operators. That is, they should be positive maps, as defined below:

Definition 3 (Positive Map)**.** *A linear map $\mathcal{M} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is positive if $\mathcal{M}(X_A)$ is positive semi-definite for all positive semi-definite $X_A \in \mathcal{B}(\mathcal{H}_A)$.*

If we were dealing with classical systems, then positivity would be sufficient to describe the class of physical evolutions. However, above we argued that we are working in the "black box" picture of quantum physical evolutions, and here, in principle, we allow for Alice to prepare the input system A to be one share of an arbitrary two-party state $\rho_{RA} \in \mathcal{D}(\mathcal{H}_R \otimes \mathcal{H}_A)$, where R is a reference system of arbitrary size. So this means that the evolution consisting of the identity acting on the reference system R and the map $\mathcal{N}$ acting on system A should take ρ_{RA} to a density operator on systems R and B. Let $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}$ denote this evolution, where id_R denotes the identity superoperator acting on the system R.

How do we describe the evolution $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}$ mathematically? Let X_{RA} be an arbitrary operator acting on $\mathcal{H}_R \otimes \mathcal{H}_A$, and let $\{|i\rangle_R\}$ be an orthonormal basis for $\mathcal{H}_R$. Then we can expand X_{RA} with respect to this basis as follows:

$$X_{RA} = \sum_{i,j} |i\rangle\langle j|_R \otimes X_A^{i,j}, \tag{3}$$

and the action of $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}$ on X_{RA} (for linear $\mathcal{N}$) is defined as follows:

$$\begin{aligned}
(\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(X_{RA}) &= (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})\left(\sum_{i,j} |i\rangle\langle j|_R \otimes X_A^{i,j}\right) && (4)\\
&= \sum_{i,j} (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})\left(|i\rangle\langle j|_R \otimes X_A^{i,j}\right) && (5)\\
&= \sum_{i,j} \mathrm{id}_R\left(|i\rangle\langle j|_R\right) \otimes \mathcal{N}_{A\to B}\left(X_A^{i,j}\right) && (6)\\
&= \sum_{i,j} |i\rangle\langle j|_R \otimes \mathcal{N}_{A\to B}\left(X_A^{i,j}\right). && (7)
\end{aligned}$$

That is, the identity superoperator id_R has no effect on the R system. The above development leads to the notion of a linear map being *completely positive* and our next criterion for any quantum physical evolution:

Definition 4 (Completely Positive Map)**.** *A linear map $\mathcal{M} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is completely positive if $\mathrm{id}_R \otimes \mathcal{M}$ is a positive map for a reference system R of arbitrary size.*

Criterion 5 (Complete Positivity)**.** *A quantum channel is a completely positive map.*

There is one last requirement that we impose for quantum physical evolutions, known as *trace preservation*. This requirement again stems from the reasonable constraint that $\mathcal{N}$ should map density operators to density operators. That is, it should be the case that $\mathrm{Tr}\{\rho_A\} = \mathrm{Tr}\{\mathcal{N}(\rho_A)\} = 1$ for all input density operators ρ_A. However, now that have argued for linearity of every quantum physical evolution, trace preservation on density operators combined with linearity implies that quantum channels are trace preserving on the set of all operators. This is due to the fact that there are sets of density operators that form a basis for $\mathcal{B}(\mathcal{H}_A)$. Indeed, one such basis of density operators is as follows:

$$\rho_A^{x,y} = \begin{cases} |x\rangle\langle x|_A & \text{if } x = y \\ \frac{1}{2}\left(|x\rangle_A + |y\rangle_A\right)\left(\langle x|_A + \langle y|_A\right) & \text{if } x < y \\ \frac{1}{2}\left(|x\rangle_A + i|y\rangle_A\right)\left(\langle x|_A - i\langle y|_A\right) & \text{if } x > y \end{cases}. \tag{8}$$

Consider that for all x, y such that $x < y$, the following holds

$$|x\rangle\langle y|_A = \left(\rho_A^{x,y} - \frac{1}{2}\rho_A^{x,x} - \frac{1}{2}\rho_A^{y,y}\right) - i\left(\rho_A^{y,x} - \frac{1}{2}\rho_A^{x,x} - \frac{1}{2}\rho_A^{y,y}\right), \tag{9}$$

$$|y\rangle\langle x|_A = \left(\rho_A^{x,y} - \frac{1}{2}\rho_A^{x,x} - \frac{1}{2}\rho_A^{y,y}\right) + i\left(\rho_A^{y,x} - \frac{1}{2}\rho_A^{x,x} - \frac{1}{2}\rho_A^{y,y}\right), \tag{10}$$

so that we can represent any operator X_A as a linear combination of density operators from the set $\left\{\rho_A^{x,y}\right\}$. This leads to our final criterion for quantum channels:

Criterion 6 (Trace Preservation)**.** *A quantum channel is trace preserving, in the sense that* $\text{Tr}\{X_A\} = \text{Tr}\{\mathcal{N}(X_A)\}$ *for all* $X_A \in \mathcal{B}(\mathcal{H}_A)$.

Definition 7 (Quantum Channel)**.** *A quantum channel is a linear, completely positive, trace preserving map, corresponding to a quantum physical evolution.*

Criteria 2, 5, and 6 detailed above lead naturally to the Choi-Kraus representation theorem, which states that a map satisfies all three criteria if and only if it takes a particular form according to a Choi-Kraus decomposition:

Theorem 8 (Choi-Kraus)**.** *A map* $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ *(denoted also by* $\mathcal{N}_{A\to B}$*) is linear, completely positive, and trace-preserving if and only if it has a Choi-Kraus decomposition as follows:*

$$\mathcal{N}_{A\to B}(X_A) = \sum_{l=0}^{d-1} V_l X_A V_l^\dagger, \tag{11}$$

where $X_A \in \mathcal{B}(\mathcal{H}_A)$*,* $V_l \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_B)$ *for all* $l \in \{0, \ldots, d-1\}$*,*

$$\sum_{l=0}^{d-1} V_l^\dagger V_l = I_A, \tag{12}$$

and d *need not be any larger than* $\dim(\mathcal{H}_A)\dim(\mathcal{H}_B)$.

Before we delve into a proof, it is helpful to give a sketch. There is an easier part and a more challenging part of the proof. For the more challenging part, a helpful tool is an operator called the Choi operator:

Definition 9 (Choi Operator)**.** *Let* $\mathcal{H}_R$ *and* $\mathcal{H}_A$ *be isomorphic Hilbert spaces, and let* $\{|i\rangle_R\}$ *and* $\{|i\rangle_A\}$ *be orthonormal bases for* $\mathcal{H}_R$ *and* $\mathcal{H}_A$*, respectively. Let* $\mathcal{H}_B$ *be some other Hilbert space, and let* $\mathcal{N} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ *be a linear map (abbreviated as* $\mathcal{N}_{A\to B}$*). The Choi operator corresponding to* $\mathcal{N}_{A\to B}$ *and the bases* $\{|i\rangle_R\}$ *and* $\{|i\rangle_A\}$ *is defined as the following operator:*

$$(\text{id}_R \otimes \mathcal{N}_{A\to B}) (|\Gamma\rangle\langle\Gamma|_{RA}) = \sum_{i,j=0}^{d_A-1} |i\rangle\langle j|_R \otimes \mathcal{N}_{A\to B}(|i\rangle\langle j|_A), \tag{13}$$

where $d_A \equiv \dim(\mathcal{H}_A)$ *and* $|\Gamma\rangle_{RA}$ *is an unnormalized maximally entangled vector, as defined in (**??**):*

$$|\Gamma\rangle_{RA} \equiv \sum_{i=0}^{d_A-1} |i\rangle_R \otimes |i\rangle_A. \tag{14}$$

If $\mathcal{N}_{A\to B}$ is a completely positive map, then the Choi operator is positive semi-definite. This follows as a direct consequence of Definition 4 and the fact that $|\Gamma\rangle\langle\Gamma|_{RA}$ is positive semi-definite. The converse is true as well, and Exercise 11 asks you to verify this. The converse is in some sense a much more powerful statement. Definition 4 suggests that we would have to check a seemingly infinite number of cases in order to verify whether a given linear map is completely positive, but the converse statement establishes that we need to check only one condition: whether the Choi operator is positive semi-definite.

Why else is the Choi operator a useful tool? One other important reason is that it encodes how a quantum channel acts on any possible input operator X_A, and thus specifies the channel completely. Consider that we can expand the Choi operator as a matrix of matrices (of total size $d_A d_B \times d_A d_B$) in the following way, by exploiting properties of the tensor product:

$$\begin{bmatrix} \mathcal{N}_{A\to B}(|0\rangle\langle 0|_A) & \mathcal{N}_{A\to B}(|0\rangle\langle 1|_A) & \cdots & \mathcal{N}_{A\to B}(|0\rangle\langle d_A - 1|_A) \\ \mathcal{N}_{A\to B}(|1\rangle\langle 0|_A) & \mathcal{N}_{A\to B}(|1\rangle\langle 1|_A) & \cdots & \mathcal{N}_{A\to B}(|1\rangle\langle d_A - 1|_A) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{N}_{A\to B}(|d_A - 1\rangle\langle 0|_A) & \mathcal{N}_{A\to B}(|d_A - 1\rangle\langle 1|_A) & \cdots & \mathcal{N}_{A\to B}(|d_A - 1\rangle\langle d_A - 1|_A) \end{bmatrix}. \tag{15}$$

So if we would like to figure out how the channel $\mathcal{N}_{A\to B}$ acts on an input operator X_A, we can first expand X_A with respect to the orthonormal basis $\{|i\rangle_A\}$ as $X_A = \sum_{i,j} x^{i,j}|i\rangle\langle j|_A$ and then apply the channel, using linearity:

$$\mathcal{N}_{A\to B}(X_A) = \mathcal{N}_{A\to B}\left(\sum_{i,j} x^{i,j}|i\rangle\langle j|_A\right) = \sum_{i,j} x^{i,j}\mathcal{N}_{A\to B}(|i\rangle\langle j|_A). \tag{16}$$

So the procedure is to expand X_A as above, multiple the (i,j) coefficient $x^{i,j}$ with the (i,j) entry in the Choi operator, and then sum these operators over all indices i and j.

Proof of Theorem 8. We first prove the easier "if-part" of the theorem. So let us suppose that $\mathcal{N}_{A\to B}$ has the form in (11) and that the condition in (12) holds as well. Then $\mathcal{N}_{A\to B}$ is clearly a linear map. It is completely positive because $(\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(X_{RA}) \geq 0$ if $X_{RA} \geq 0$ when $\mathcal{N}_{A\to B}$ has the form in (11), and this holds for a reference system R of arbitrary size. That is, consider from (7) that $\{I_R \otimes V_l\}$ is a set of Kraus operators for the extended channel $\mathrm{id}_R \otimes \mathcal{N}_{A\to B}$ and thus

$$\begin{aligned} (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(X_{RA}) &= \sum_{l=0}^{d-1} (I_R \otimes V_l) X_{RA} (I_R \otimes V_l^\dagger) && (17) \\ &= \sum_{l=0}^{d-1} (I_R \otimes V_l) X_{RA} (I_R \otimes V_l)^\dagger. && (18) \end{aligned}$$

We know that $(I_R \otimes V_l) X_{RA} (I_R \otimes V_l)^\dagger \geq 0$ for all l when $X_{RA} \geq 0$, and the same is true for the sum. Trace preservation follows because

$$\begin{aligned} \mathrm{Tr}\{\mathcal{N}_{A\to B}(X_A)\} &= \mathrm{Tr}\left\{\sum_{l=0}^{d-1} V_l X_A V_l^\dagger\right\} && (19) \\ &= \mathrm{Tr}\left\{\sum_{l=0}^{d-1} V_l^\dagger V_l X_A\right\} && (20) \\ &= \mathrm{Tr}\{X_A\}, && (21) \end{aligned}$$

where the second line is from linearity and cyclicity of trace and the last line follows from the condition in (12).

We now prove the more difficult "only-if" part. Let $d_A \equiv \dim(\mathcal{H}_A)$ and $d_B \equiv \dim(\mathcal{H}_B)$. Consider that we can diagonalize the Choi operator as given in Definition 9, because it is positive semi-definite:

$$\mathcal{N}_{A\to B}\left(|\Gamma\rangle\langle\Gamma|_{RA}\right) = \sum_{l=0}^{d-1} |\phi_l\rangle\langle\phi_l|_{RB}\,, \tag{22}$$

where $d \leq d_A d_B$ is the Choi rank of the map $\mathcal{N}_{A\to B}$. (This decomposition does not necessarily have to be such that the vectors $\{|\phi_l\rangle_{RB}\}$ are orthonormal, but keep in mind that there is always a choice such that $d \leq d_A d_B$.) Consider by inspecting (13) that

$$\left(\langle i|_R \otimes I_B\right)\left(\mathcal{N}_{A\to B}\left(|\Gamma\rangle\langle\Gamma|_{RA}\right)\right)\left(|j\rangle_R \otimes I_B\right) = \mathcal{N}_{A\to B}\left(|i\rangle\langle j|\right). \tag{23}$$

Now, consider that for any bipartite vector $|\phi\rangle_{RB}$, we can expand it in terms of an orthonormal basis $\{|j\rangle_B\}$ and the basis $\{|i\rangle_R\}$ given above:

$$|\phi\rangle_{RB} = \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \alpha_{ij}|i\rangle_R \otimes |j\rangle_B. \tag{24}$$

Let $V_{A\to B}$ denote the following linear operator:

$$V_{A\to B} \equiv \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \alpha_{i,j}|j\rangle_B\langle i|_A, \tag{25}$$

where $\{|i\rangle_A\}$ is the orthonormal basis given above. Then we see that

$$\begin{aligned}
\left(I_R \otimes V_{A\to B}\right)|\Gamma\rangle_{RA} &= \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \alpha_{i,j}\,|j\rangle_B\,\langle i|_A \sum_{k=0}^{d_A-1} |k\rangle_R \otimes |k\rangle_A && (26)\\
&= \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1}\sum_{k=0}^{d_A-1} \alpha_{i,j}|k\rangle_R \otimes |j\rangle_B\,\langle i|k\rangle_A && (27)\\
&= \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \alpha_{ij}\,|i\rangle_R \otimes |j\rangle_B && (28)\\
&= |\phi\rangle_{RB}\,. && (29)
\end{aligned}$$

So this means that for all bipartite vectors $|\phi\rangle_{RB}$, we can find a linear operator $V_{A\to B}$ such that $\left(I_R \otimes V_{A\to B}\right)|\Gamma\rangle_{RA} = |\phi\rangle_{RB}$. Consider also that

$$\begin{aligned}
\langle i|_R\,|\phi\rangle_{RB} &= \langle i|_R\left(I_R \otimes V_{A\to B}\right)|\Gamma\rangle_{RA} && (30)\\
&= V_{A\to B}|i\rangle_A. && (31)
\end{aligned}$$

Applying this to our case of interest, for each l, we can write

$$|\phi_l\rangle_{RB} = I_R \otimes \left(V_l\right)_{A\to B}|\Gamma\rangle_{RA}\,, \tag{32}$$

where $(V_l)_{A\to B}$ is some linear operator of the form in (25). After making this observation, we realize that it is possible to write

$$\mathcal{N}_{A\to B}(|i\rangle\langle j|) = (\langle i|_R \otimes I_B)(\mathcal{N}_{A\to B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) \tag{33}$$

$$= (\langle i|_R \otimes I_B)\sum_{l=0}^{d-1}|\phi_l\rangle\langle\phi_l|_{RB}(|j\rangle_R \otimes I_B) \tag{34}$$

$$= \sum_{l=0}^{d-1}[(\langle i|_R \otimes I_B)|\phi_l\rangle_{RB}][\langle\phi_l|_{RB}(|j\rangle_R \otimes I_B)] \tag{35}$$

$$= \sum_{l=0}^{d-1} V_l|i\rangle\langle j|_A V_l^\dagger. \tag{36}$$

By linearity of the map $\mathcal{N}_{A\to B}$, exploiting the above result, and the development in (16), it follows that the action of $\mathcal{N}_{A\to B}$ on any input operator X_A can be written as follows:

$$\mathcal{N}_{A\to B}(X_A) = \sum_{l=0}^{d-1} V_l X_A V_l^\dagger. \tag{37}$$

To prove the condition in (12), let us begin by exploiting the fact that the map $\mathcal{N}_{A\to B}$ is trace preserving, so that

$$\mathrm{Tr}\{\mathcal{N}_{A\to B}(|i\rangle\langle j|_A)\} = \mathrm{Tr}\{|i\rangle\langle j|_A\} = \delta_{ij}. \tag{38}$$

for all operators $\{|i\rangle\langle j|_A\}_{i,j}$. But consider also that

$$\mathrm{Tr}\{\mathcal{N}_{A\to B}(|i\rangle\langle j|_A)\} = \mathrm{Tr}\left\{\sum_l V_l(|i\rangle\langle j|_A)V_l^\dagger\right\} \tag{39}$$

$$= \mathrm{Tr}\left\{\sum_l V_l^\dagger V_l(|i\rangle\langle j|_A)\right\} \tag{40}$$

$$= \langle j|_A \sum_l V_l^\dagger V_l\,|i\rangle_A. \tag{41}$$

Thus, in order to have consistency with (38), we require that $\langle j|_A \sum_l V_l^\dagger V_l|i\rangle_A = \delta_{i,j}$, or equivalently, for (12) to hold. □

Remark 10. *If the decomposition in* (22) *is a spectral decomposition, then it follows that the Kraus operators* $\{V_l\}$ *are orthogonal with respect to the Hilbert–Schmidt inner product:*

$$\mathrm{Tr}\left\{V_l^\dagger V_k\right\} = \mathrm{Tr}\left\{V_l^\dagger V_l\right\}\delta_{l,k}. \tag{42}$$

This follows from the fact that

$$\delta_{l,k}\langle\phi_l|\phi_l\rangle = \langle\phi_l|\phi_k\rangle \tag{43}$$

$$= \langle\Gamma|_{RB}\left[I_R \otimes \left(V_l^\dagger V_k\right)_B\right]|\Gamma\rangle_{RB} \tag{44}$$

$$= \mathrm{Tr}\left\{V_l^\dagger V_k\right\}, \tag{45}$$

where in the third line we have applied the result of Exercise **??***.*

Exercise 11. *Prove that a linear map $\mathcal{N}$ is completely positive if its corresponding Choi operator, as defined in Definition 9, is a positive semi-definite operator. (Hint: Use the fact that any positive semi-definite operator can be diagonalized, the fact that* $\mathrm{id}_R \otimes \mathcal{N}$ *is linear, and use something similar to* (26)–(29)*).*

3 Unique Linear Extension of a Quantum Physical Evolution

Recall in Section 2 that we argued on physical grounds how any quantum physical evolution $\mathcal{N}$ should be convex linear when acting on the space $\mathcal{D}(\mathcal{H}_A)$ of density operators:

$$\mathcal{N}(\lambda\rho_A + (1-\lambda)\sigma_A) = \lambda\mathcal{N}(\rho_A) + (1-\lambda)\mathcal{N}(\sigma_A), \tag{46}$$

where $\rho_A, \sigma_A \in \mathcal{D}(\mathcal{H}_A)$ and $\lambda \in [0,1]$. Here we show how to construct a unique linear extension $\widetilde{\mathcal{N}}$ of $\mathcal{N}$, whose action is well defined on the space of all operators $X_A \in \mathcal{B}(\mathcal{H}_A)$. The development follows the approach given in Proposition 2.30 of "The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement," by Teiko Heinosaari and Mario Ziman.

We first define $\widetilde{\mathcal{N}}(0) \equiv 0$, where the inputs and outputs are understood to be the zero operator. We next extend the action of $\mathcal{N}$ to all positive semi-definite operators $P_A \neq 0$ as follows:

$$\widetilde{\mathcal{N}}(P_A) \equiv \mathrm{Tr}\{P_A\}\mathcal{N}([\mathrm{Tr}\{P_A\}]^{-1}\, P_A), \tag{47}$$

where it is clear that this is well defined from $\mathcal{N}$ because $[\mathrm{Tr}\{P_A\}]^{-1}\, P_A$ is a density operator. Now consider for a constant $s > 0$ that we have scale invariance:

$$\begin{aligned}\widetilde{\mathcal{N}}(sP_A) &= \mathrm{Tr}\{sP_A\}\mathcal{N}([\mathrm{Tr}\{sP_A\}]^{-1}\, sP_A) &(48)\\ &= s\,\mathrm{Tr}\{P_A\}\mathcal{N}([\mathrm{Tr}\{P_A\}]^{-1}\, P_A) &(49)\\ &= s\widetilde{\mathcal{N}}(P_A). &(50)\end{aligned}$$

Furthermore, for two non-zero positive semi-definite operators P_A and Q_A, we have the following additivity relation:

$$\widetilde{\mathcal{N}}(P_A + Q_A) = \widetilde{\mathcal{N}}(P_A) + \widetilde{\mathcal{N}}(Q_A), \tag{51}$$

which follows because

$$\begin{aligned}&\widetilde{\mathcal{N}}(P_A + Q_A)\\ &= \mathrm{Tr}\{P_A + Q_A\}\mathcal{N}([\mathrm{Tr}\{P_A + Q_A\}]^{-1}\,(P_A + Q_A)) &(52)\\ &= \mathrm{Tr}\{P_A + Q_A\}\mathcal{N}\left(\frac{1}{\mathrm{Tr}\{P_A + Q_A\}}P_A + \frac{1}{\mathrm{Tr}\{P_A + Q_A\}}Q_A\right) &(53)\\ &= \mathrm{Tr}\{P_A + Q_A\}\mathcal{N}\left(\frac{\mathrm{Tr}\{P_A\}}{\mathrm{Tr}\{P_A + Q_A\}}\frac{P_A}{\mathrm{Tr}\{P_A\}} + \frac{\mathrm{Tr}\{Q_A\}}{\mathrm{Tr}\{P_A + Q_A\}}\frac{Q_A}{\mathrm{Tr}\{Q_A\}}\right) &(54)\\ &= \mathrm{Tr}\{P_A\}\mathcal{N}\left(\frac{P_A}{\mathrm{Tr}\{P_A\}}\right) + \mathrm{Tr}\{Q_A\}\mathcal{N}\left(\frac{Q_A}{\mathrm{Tr}\{Q_A\}}\right) &(55)\\ &= \widetilde{\mathcal{N}}(P_A) + \widetilde{\mathcal{N}}(Q_A), &(56)\end{aligned}$$

where in the fourth equality, we exploited convex linearity of the quantum physical evolution $\mathcal{N}$.

For the next step, recall that any Hermitian operator T_A can be written as a linear combination of a positive part and a negative part: $T_A = T_A^+ - T_A^-$, where both T_A^+ and T_A^- are positive semi-definite operators. So we define the action of $\widetilde{\mathcal{N}}$ on any Hermitian operator T_A as follows:

$$\widetilde{\mathcal{N}}(T_A) \equiv \widetilde{\mathcal{N}}(T_A^+) - \widetilde{\mathcal{N}}(T_A^-). \tag{57}$$

To see that the following additivity relation holds for all Hermitian S_A and T_A

$$\widetilde{\mathcal{N}}(S_A + T_A) = \widetilde{\mathcal{N}}(S_A) + \widetilde{\mathcal{N}}(T_A), \tag{58}$$

consider that

$$S_A + T_A = (S_A + T_A)^+ - (S_A + T_A)^-, \tag{59}$$

while also

$$S_A + T_A = S_A^+ + T_A^+ - S_A^- - T_A^-. \tag{60}$$

Equating both sides, we find that

$$(S_A + T_A)^+ + S_A^- + T_A^- = (S_A + T_A)^- + S_A^+ + T_A^+. \tag{61}$$

Now we exploit this equality, (51), and the definition in (57) to establish (58).

The final step is to extend the action of $\widetilde{\mathcal{N}}$ to all operators $X_A \in \mathcal{B}(\mathcal{H}_A)$. Here, we recall that any linear operator can be written in terms of a real and imaginary part as follows:

$$X_A^R \equiv \frac{1}{2}\left(X_A + X_A^\dagger\right), \qquad X_A^I \equiv \frac{1}{2i}\left(X_A - X_A^\dagger\right), \tag{62}$$

where by inspection, X_A^R and X_A^I are Hermitian operators. So we define

$$\widetilde{\mathcal{N}}(X_A) \equiv \widetilde{\mathcal{N}}(X_A^R) + i\widetilde{\mathcal{N}}(X_A^I). \tag{63}$$

This completes the development of a well defined linear extension $\widetilde{\mathcal{N}}$ of the quantum physical evolution $\mathcal{N}$.

To show that it is unique, recall that any operator X_A can be expanded as a linear combination of density operators from the basis $\{\rho_A^{x,y}\}$, defined in (8), as follows:

$$X_A = \sum_{x,y} \alpha_{x,y} \rho_A^{x,y}, \tag{64}$$

where $\alpha_{x,y} \in \mathbb{C}$ for all x and y. It is straightforward to show from the above development that

$$\widetilde{\mathcal{N}}(X_A) = \sum_{x,y} \alpha_{x,y} \mathcal{N}(\rho_A^{x,y}). \tag{65}$$

Now suppose that $\mathcal{N}'$ is some other linear map for which $\mathcal{N}'(\rho_A) = \mathcal{N}(\rho_A)$ for all $\rho_A \in \mathcal{D}(\mathcal{H}_A)$. Then the following equality holds for all $X_A \in \mathcal{B}(\mathcal{H}_A)$:

$$\mathcal{N}'(X_A) = \sum_{x,y} \alpha_{x,y} \mathcal{N}'(\rho_A^{x,y}) = \sum_{x,y} \alpha_{x,y} \mathcal{N}(\rho_A^{x,y}) = \widetilde{\mathcal{N}}(X_A). \tag{66}$$

As a result, $\mathcal{N}' = \widetilde{\mathcal{N}}$, given that they have the same action on every operator $X_A \in \mathcal{B}(\mathcal{H}_A)$.