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Lecture 4

- can never have perfect knowledge of a state
- errors can also occur in preparation, evolution, or measurement
- relax this assumption → "noisy quantum theory" subsumes probability theory & noiseless quantum theory

Proceed in the following order:

1) density operators
2) general form of measurements
3) composite noisy systems
4) noisy evolution

Noisy States

Suppose a third party prepares a state $|\psi_x\rangle$ w/ prob. $p(x)$, but doesn't tell us which one he prepared
- our best description is as ensemble

$$\mathcal{E} \equiv \{p(x), |\psi_x\rangle\}$$

What is the outcome of a measurement w/ projectors $\{\Pi_j\}$ such that $\sum_j \Pi_j = I$?

Let J denote R.V. for measurement outcome

~~Suppose that state is $|\psi_x\rangle$.~~

Then conditional probability for getting outcome j is

$$p_{J|X}(j|x) = \langle\psi_x|\Pi_j|\psi_x\rangle$$

& post-measurement state is

$$\frac{\Pi_j|\psi_x\rangle}{\sqrt{p(j|x)}}$$

But, since we don't know x, the relevant prob. for measurement outcome is

<u>unconditional</u> prob. $p_J(j)$

From law of total prob.,

$$p_J(j) = \sum_x p_{J|X}(j|x)\, p_X(x)$$

$$= \sum_x \langle\psi_x|\Pi_j|\psi_x\rangle\, p_X(x)$$

Define the trace of operator A as

$$\mathrm{Tr}\{A\} \equiv \sum_i \langle i|A|i\rangle \quad \text{where } \{|i\rangle\} \text{ o.n. basis.}$$

Then

$$\mathrm{Tr}\{\Pi_j|\psi_x\rangle\langle\psi_x|\} = \sum_i \langle i|\Pi_j|\psi_x\rangle\langle\psi_x|i\rangle$$

$$= \sum_i \langle\psi_x|i\rangle\langle i|\Pi_j|\psi_x\rangle$$

$$= \langle\psi_x|\left(\sum_i |i\rangle\langle i|\right)\Pi_j|\psi_x\rangle$$

$$= \langle\psi_x|\Pi_j|\psi_x\rangle$$

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Then $p_J(j) = \sum_x \text{Tr}\{\Pi_j |\psi_x\rangle\langle\psi_x|\} p_X(x)$

$$= \text{Tr}\{\Pi_j \left(\sum_x p_X(x) |\psi_x\rangle\langle\psi_x|\right)\}$$

rewrite as

$$p_J(j) = \text{Tr}\{\Pi_j \rho\}$$

where ρ is density operator

$$\rho \equiv \sum_x p_X(x) |\psi_x\rangle\langle\psi_x|$$

AKA expected density operator for ensemble

$$\rho = \mathbb{E}_X \{|\psi_X\rangle\langle\psi_X|\}$$

Properties of density operator

1) unit trace, 2) positive 3) Hermitian

1) $\text{Tr}\{\rho\} = \text{Tr}\{\sum_x p(x) |\psi_x\rangle\langle\psi_x|\}$

$$= \sum_x p(x) \text{Tr}\{|\psi_x\rangle\langle\psi_x|\}$$

$$= \sum_x p(x) \langle\psi_x|\psi_x\rangle$$

$$= \sum_x p(x) = 1$$

2) $\forall |\varphi\rangle \quad \langle\varphi|\rho|\varphi\rangle \geq 0$

$$\langle\varphi|\rho|\varphi\rangle = \langle\varphi|\left(\sum_x p(x) |\psi_x\rangle\langle\psi_x|\right)|\varphi\rangle$$

$$= \sum_x p(x) \langle\varphi|\psi_x\rangle\langle\psi_x|\varphi\rangle$$

$$= \sum_x p(x) |\langle\psi_x|\varphi\rangle|^2 \geq 0$$

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3) $\rho^\dagger = \left(\sum_x p(x) |\psi_x\rangle\langle\psi_x| \right)^\dagger$

$= \sum_x p(x) \left(|\psi_x\rangle\langle\psi_x| \right)^\dagger$

$= \sum_x p(x) |\psi_x\rangle\langle\psi_x|$

$= \rho$

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every ensemble has a unique density operator but every density operator does not correspond to a unique ensemble.

e.g. $\left\{ \left\{ \frac{1}{2}, |0\rangle \right\}, \left\{ \frac{1}{2}, |1\rangle \right\} \right\}$

& $\left\{ \left\{ \frac{1}{2}, |+\rangle \right\}, \left\{ \frac{1}{2}, |-\rangle \right\} \right\}$

have same density operator

$\frac{I}{2}$ (maximally mixed state)

In spite of this, there is a "canonical" ensemble for a given density operator (though still not unique)

since every density ~~operator~~ ρ is Hermitian, can diagonalize it

$$\rho = \sum_{x=0}^{d-1} \lambda_x |\phi_x\rangle\langle\phi_x|$$

eigenvals (probabilities) — O.N. eigenvectors

ensemble is then

$\{ \lambda_x, |\phi_x\rangle \}$

revise the 1st postulate: a quantum state specified by ρ.

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(5)

Noiseless Evolution of an ensemble

Suppose some ensemble $\{p(x), |\psi_x\rangle\}$

suppose we know the state $|\psi_x\rangle$

Then after evolution U, new state is

$$U|\psi_x\rangle$$

can say that we have a new ensemble

$$\{p(x), U|\psi_x\rangle\}$$

density operator for original ensemble is

$$\rho = \sum_x p(x) |\psi_x\rangle\langle\psi_x|$$

density operator for evolved ensemble is

$$\sum_x p(x) U|\psi_x\rangle\langle\psi_x|U^\dagger = U\left(\sum_x p(x)|\psi_x\rangle\langle\psi_x|\right)U^\dagger$$

revise postulate II $= \boxed{U\rho U^\dagger}$

evolution of the density operator

Noiseless Measurement

have ensemble $\{p(x), |\psi_x\rangle\}$ again

Suppose for now that we know the state is $|\psi_x\rangle$

& we perform a measurement $\{\Pi_j\}$

probability for getting j is

$$p_{J|X}(j|x) = \langle\psi_x|\Pi_j|\psi_x\rangle$$ & post measurement state is

$$\frac{\Pi_j|\psi_x\rangle}{\sqrt{p_{J|X}(j|x)}}$$

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Suppose that we perform measurement, but we don't know on which state ~~we performed the measurement~~ (though we know measurement result) 6

ensemble is then

$$\mathcal{E}_j = \left\{ p_{X|J}(x|j), \frac{\Pi_j|\psi_x\rangle}{\sqrt{p_{J|X}(j|x)}} \right\}$$

we know;

density operator of ensemble is

$$\sum_x p(x|j) \frac{\Pi_j|\psi_x\rangle\langle\psi_x|\Pi_j}{p_{J|X}(j|x)}$$

$$= \Pi_j \left(\sum_x \frac{p(x|j)}{p(j|x)} |\psi_x\rangle\langle\psi_x| \right) \Pi_j$$

(Note: $p(x|j) = \frac{p(j|x)}{p(j)}$

$$\therefore \quad = \Pi_j \left(\sum_x \frac{p(j|x)p(x)}{p(j|x)p(j)} |\psi_x\rangle\langle\psi_x| \right) \Pi_j$$

$$= \Pi_j \frac{\left(\sum_x p(x)|\psi_x\rangle\langle\psi_x|\right)}{p(j)} \Pi_j$$

$$= \boxed{\frac{\Pi_j \rho \Pi_j}{p(j)}}$$

← this is how the density operator evolves under measurement

use law of total probability to get $p(j)$

$$p_J(j) = \sum_x p_{J|X}(j|x)\, p(x)$$

$$= \sum_x \langle\psi_x|\Pi_j|\psi_x\rangle\, p(x)$$

$$= \sum_x \mathrm{Tr}\{|\psi_x\rangle\langle\psi_x|\Pi_j\} p(x)$$

$$= \mathrm{Tr}\left\{\left(\sum_x p(x)|\psi_x\rangle\langle\psi_x|\right)\Pi_j\right\}$$

$$= \mathrm{Tr}\{\rho\Pi_j\}$$

measures the "shadow" of ρ on Π_j subspace

1/20/2011 ~~intuition helpful,~~

~~but it all breaks down w/ a little~~
~~non-orthogonality!~~
~~(play w/ $|0\rangle$ + $|+\rangle$, for example)~~

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projectors not the most general form of measurement

- in general, can be set of operators $\{M_j\}$ such that

$$\sum_j M_j^\dagger M_j = I$$

for pure states, probabilities from measurement are

$$p(j) = \langle\psi| M_j^\dagger M_j |\psi\rangle$$

+ post-measurement state is

$$\frac{M_j|\psi\rangle}{\sqrt{p(j)}}$$

for mixed states,

$$p(j) = \mathrm{tr}\{M_j^\dagger M_j \rho\}$$

+ post-measurement state is

$$\frac{M_j \rho M_j^\dagger}{p(j)}$$

revise postulate IV.

POVM Formalism

useful if we are just interested in classical outcome of measurement, not post-measurement state

(application in classical communication over a quantum channel)

POVM - (positive operator-valued measure)

some operators $\{\underbrace{\Lambda_j}_{\text{POVM elements}}\}$

$\forall j \quad \Lambda_j \geq 0$

$\sum_j \Lambda_j = I$ ("just like" probabilities)

If state is ρ, probability for getting outcome j is

$\mathrm{Tr}\{\Lambda_j \rho\}$

Multiparty quantum states

for a tensor product Hilbert space, we have a general density operator ρ_{AB} which acts on $\mathcal{H}_A \otimes \mathcal{H}_B$

We have $\rho_{AB} \geq 0$ & $\mathrm{Tr}\{\rho_{AB}\} = 1$

so we can write

$$\rho_{AB} = \sum_z p_Z(z) \, |\phi_z\rangle\langle\phi_z|_{AB}$$

for some O.N. pure states

$$|\phi_z\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$$

If we have O.N. bases $\{|i\rangle_A\}$, $\{|k\rangle_B\}$ then $\{|i\rangle_A \otimes |k\rangle_B\}$ is an O.N. basis for $\mathcal{H}_A \otimes \mathcal{H}_B$ & we can write

$$\rho_{AB} = \sum_{ijkl} \rho^{ijkl} \, |i\rangle\langle j|_A \otimes |k\rangle\langle l|_B$$

If Alice does not have access to Bob's system, then we would like to know how to predict the outcomes of her local measurements. So if Alice performs a measurement $\{\Lambda_A^j\}$ where $\Lambda_A^j \geq 0$ & $\sum_j \Lambda_A^j = I_A$ then the global measurement is $\Lambda_A^j \otimes I_B$ b/c Bob is doing nothing. So the probabilities are given by

$$p(j) = \mathrm{Tr}\left\{ \left(\Lambda_A^j \otimes I_B\right) \rho_{AB} \right\}$$

We would like to determine a local density op. ρ_A such that $p(j) = \mathrm{Tr}\{\Lambda_A^j \rho_A\}$

Consider that $\mathrm{Tr}\{\cdot\}$ is w/ respect to any orthonormal basis. So we can take $\{|i\rangle_A \otimes |k\rangle_B\}$ + find that

$$p(j) = \sum_{i,k} \left(\langle i|_A \otimes \langle k|_B\right)\left(\Lambda^j_A \otimes I_B\right)\cdot\left(\rho_{AB}\right)\left(|i\rangle_A \otimes |k\rangle_B\right)$$

$$= \sum_{i,k} \langle i|_A \Lambda^j_A \left(\langle k|_B \rho_{AB} |k\rangle_B\right)|i\rangle_A$$

$$= \sum_i \langle i|_A \Lambda^j_A \underbrace{\left(\sum_k \langle k|_B \rho_{AB} |k\rangle_B\right)}_{\rho_A} |i\rangle_A$$

$$= \mathrm{Tr}\{\Lambda^j_A \rho_A\}$$

this is a density operator on system A

~~So partial trace~~

So we can define partial trace as

$$\sum_k \langle k|_B \rho_{AB} |k\rangle_B = \mathrm{Tr}_B\{\rho_{AB}\}$$

analogous to marginalizing a prob. distr.

We can also define it by

$$\mathrm{Tr}_B \{ |x_1\rangle\langle y_1|_A \otimes |x_2\rangle\langle y_2|_B \}$$
$$= |x_1\rangle\langle y_1|_A \ \langle y_2|x_2\rangle$$

+ extend by linearity.

One more point:

we said that shared randomness models in the CHSH game are described by distributions

$$\sum_\lambda p(\lambda) \ p(a|x,\lambda)\, p(b|y,\lambda)$$

Quantum states of the following form admit a shared randomness model

$$\sum_\lambda p(\lambda) \ |\phi_\lambda\rangle\langle\phi_\lambda|_A \otimes |\psi_\lambda\rangle\langle\psi_\lambda|_B = \sigma_{AB}$$

b/c the resulting dist.

$$p(a,b|x,y) = \mathrm{Tr}\{ (\Pi_a^{(x)} \otimes \Pi_b^{(y)})\, \sigma_{AB} \}$$

$$= \sum_{\lambda} p(\lambda) \; \mathrm{Tr}\left\{ \left(\Pi_a^{(x)} \otimes \Pi_b^{(y)} \right) \left(|\phi_\lambda\rangle\langle\phi_\lambda|_A \otimes |\psi_\lambda\rangle\langle\psi_\lambda|_B \right) \right\}$$

$$= \sum_{\lambda} p(\lambda) \; \underbrace{\mathrm{Tr}\left\{ \Pi_a^{(x)} |\phi_\lambda\rangle\langle\phi_\lambda|_A \right\}}_{p(a|x,\lambda)} \cdot \underbrace{\mathrm{Tr}\left\{ \Pi_b^{(y)} |\psi_\lambda\rangle\langle\psi_\lambda|_B \right\}}_{p(b|y,\lambda)}$$