Lecture 2

17 JUN 2014

First discuss Bell's theorem ~~& then no-cloning theorem~~.

We will give a simple more modern approach to Bell's theorem called the "CHSH game"

main reference is

"The Device Independent Outlook on Quantum Physics"

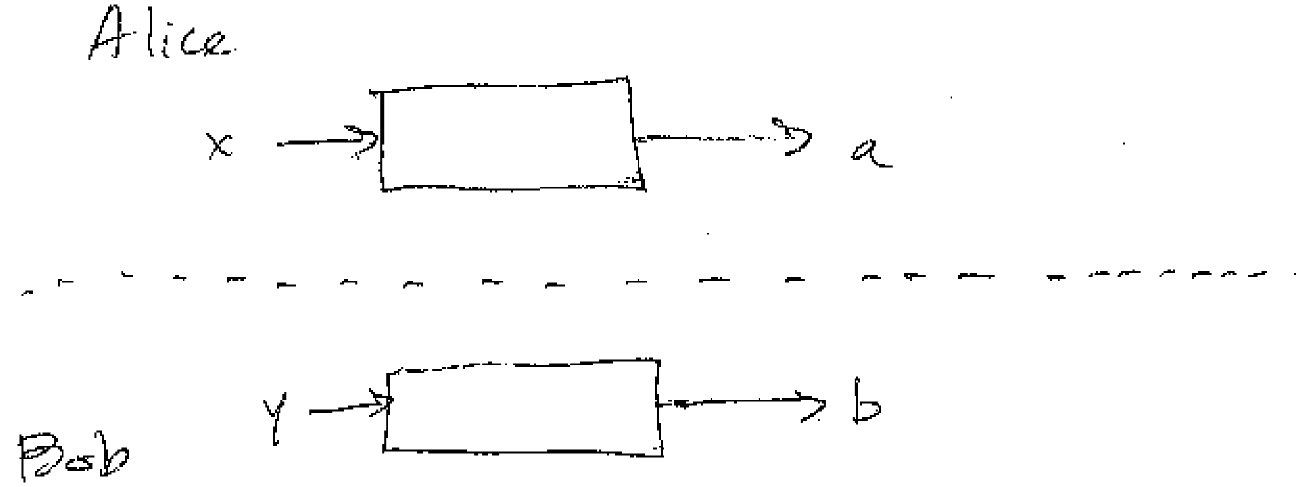
Valerio Scarani 1303.3081

- Bell's theorem is considered to be one of the greatest results in quantum mechanics & perhaps one of the first results that follows the spirit of quantum information (i.e., showing a separation between the classical & quantum theories of information)
- So many in QIT "claim" him to be one of us
- shows how entanglement is different from ~~any~~ classical notion

Bell's theorem also has practical consequences for secure communication using quantum resources.

The setup is

$$x, y, a, b \in \{0, 1\}$$



spatially separated but allowed to meet before $x + y$ are given

coins $x + y$ are tossed uniformly @ random, Alice responds w/ a + Bob w/ b.

They win if $\quad x \wedge y = a \oplus b$

We will prove that the maximum probability of winning w/ a classical strategy is $\leq 3/4$

Quantumly, one can win w/ probability $= \cos^2(\pi/8) \approx 0.85$

Proof that winning probability classically is $\leq 3/4$

In order to bound this, we need to ~~that~~ consider the distribution

$$P(a, b | x, y)$$

We will use a parameter λ to describe one's "favorite explanation" for this distribution. So by expanding w/ the law of total probability, we can write

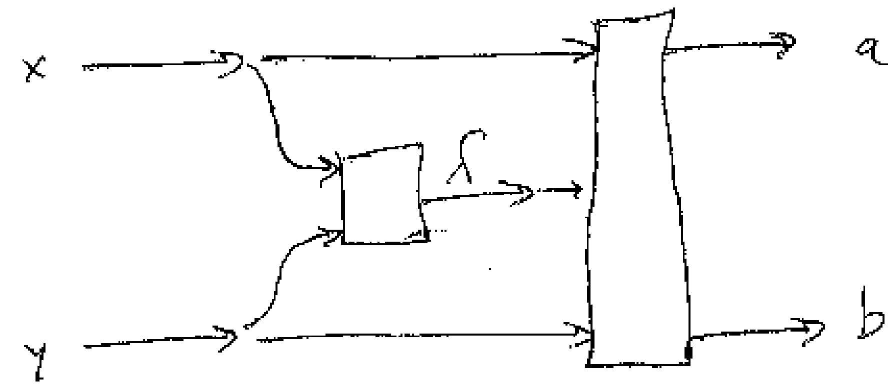
$$P(a,b|x,y) = \int d\lambda \ \rho(\lambda|x,y) \, P(a,b|x,y,\lambda)$$

where $\rho(\lambda|x,y) \geq 0$ &

$$\int d\lambda \ \rho(\lambda|x,y) = 1$$

& $P(a,b|x,y,\lambda)$ are all valid distributions

For such a general model, we can draw a picture to explain what's going on



If quantum theory is your favorite explanation, then

some bipartite quantum state

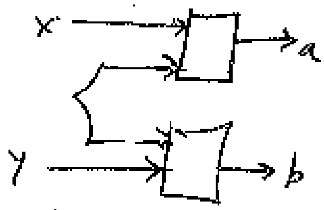
$$\rho(\lambda|x,y) = \delta(\lambda - \psi) \ \&$$

there are measurements

$\{\Pi_a^{(x)}\}_a$ & $\{\Pi_b^{(y)}\}$ such that

$$P(a,b|x,y,\lambda) = \langle\psi|\,\Pi_a^{(x)} \otimes \Pi_b^{(y)}\,|\psi\rangle$$

redraw picture as



i.e.,

$$P(a,b|x,y) = \langle\psi|\,\Pi_a^{(x)} \otimes \Pi_b^{(y)}\,|\psi\rangle$$

Idea of Bell's theorem is to put quantum theory to the test by not taking it as the favorite & seeking out alternatives

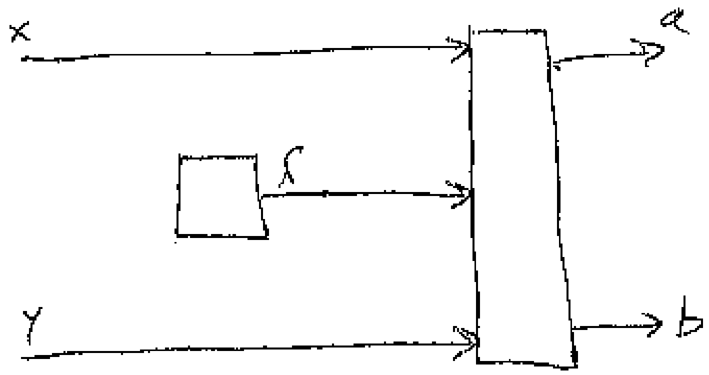
For example, we can try out classical explanations & see if they can explain the results of experiments

So we first need to formalize what we mean by a classical model

- we call such a model pre-established agreement or also "local hidden variable theory" (which sounds somewhat mysterious)

Returning to the general picture given before, there are a few things about it that are not consistent w/ our understanding of how the game works

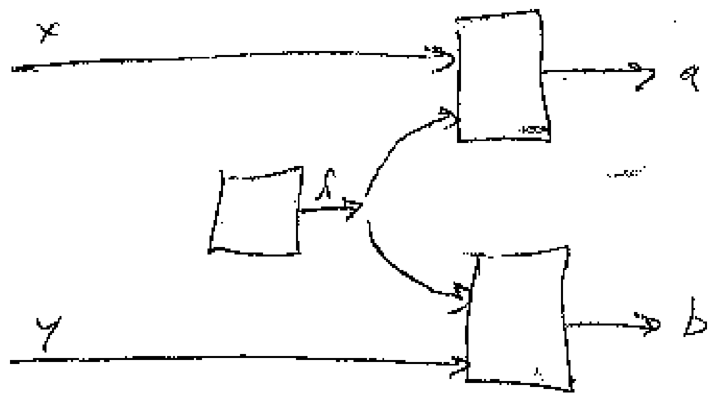
First, the parameter δ represents correlations shared between Alice & Bob before the game begins. So it is not sensible that δ depends on x & y. So we modify the picture to be as



(called "measurement independence,"

i.e., $\rho(\lambda|x,y) = \rho(\lambda)$

Next, as part of the game, we said that Alice + Bob are spatially separated, so that ~~these should be no way that~~ they should be acting locally on their inputs. The picture now changes as



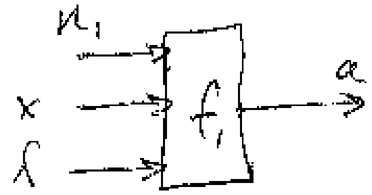
i.e., $P(a,b|x,y,\lambda) = P(a|x,\lambda)\,P(b|y,\lambda)$

so we have now argued that any classical explanation for what's going on should have the form

$$P(a,b|x,y) = \int d\lambda\, \rho(\lambda)\, P(a|x,\lambda)\, P(b|y,\lambda)$$

We now argue that it suffices to consider deterministic strategies for winning the CHSH game.

The maps $P(a|x,\lambda)$ & $P(b|y,\lambda)$ are stochastic, but we can simulate these as

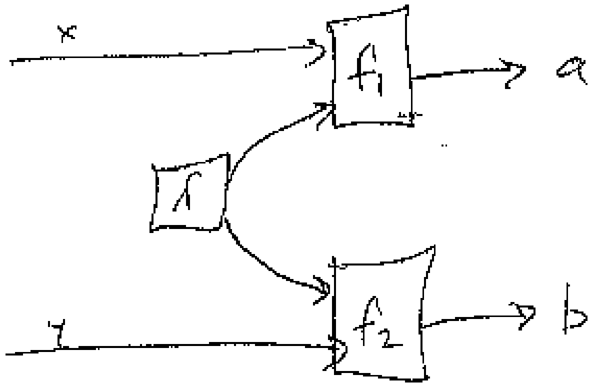


where n_1 is a local noise variable independent of x & λ & f_1 is a deterministic function, so that

$$P(a|x,\lambda) = \int dn_1 \, p(n_1) \, f(a|n_1,x,\lambda)$$

same thing for $p(b|y,\lambda)$ w/ a noise variable n_2 & function f_2

But then we can subsume the local noise variables into λ, so that the picture becomes



Furthermore, the winning probability (average) is a convex combination of the probabilities that a collection of deterministic strategies wins. Since an average of a set of numbers cannot be larger than the maximum of those numbers, $\exists$ always a deterministic strategy that does just as well as a probabilistic one.

- So it suffices to obtain an upper bound on the winning probability of any deterministic strategy.

recall that they win if $x \wedge y = a \oplus b$

consider that any deterministic strategy has $x \to a_x$, $y \to b_y$

So consider that

x	y	$x \wedge y$	a_x	b_y	$a_x \oplus b_y$
0	0	0			$a_0 \oplus b_0$
0	1	0			$a_0 \oplus b_1$
1	0	0			$a_1 \oplus b_0$
1	1	1			$a_1 \oplus b_1$
		1			0

Is it possible to win all of the time? would imply a contradiction

$1 \neq 0$

can only win 3/4 of the time.

So how is it possible to win 85%
of the time in QM?

First, we allow them to share
the maximally entangled state

$$|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$

Alice measures the observables

$$A_0 = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad A_1 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

depending on x &

Bob measures

$$B_0 = \frac{\sigma_x + \sigma_z}{\sqrt{2}} \quad \text{or} \quad B_1 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}$$

depending on y

How to measure observables?

can write $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

outcomes are +1 & -1 w/ projectors $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$

For $\sigma_x = |+\rangle\langle+| - |-\rangle\langle-|$

where $|+\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$

$|-\rangle \equiv \frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right)$

outcomes are $+1$ & -1 and projectors are

$\{|+\rangle\langle+|, |-\rangle\langle-|\}$

similar kind of thing for Bob

identify a' as the measurement outcome so that

$a' = (-1)^a$ $\quad a' \in \{+1, -1\}$

$a \in \{0, 1\}$

Consider that

$\langle\psi|_{AB}\left(A_x \otimes B_y\right)|\psi\rangle_{AB}$

is the expected value of the product of their measurement outcomes

When ~~xy~~ $xy \in \{00, 01, 10\}$

they should report the same outcome.

So in these cases,

$\langle\psi|_{AB}(A_x \otimes B_y)|\psi\rangle_{AB}$ is the prob. of winning minus prob. of losing

When ~~xy=11~~ $xy = 11$, they should report different outcomes so that winning prob. − losing prob is equal to

$$-\langle\psi|_{AB}\, A_1 \otimes B_1 |\psi\rangle_{AB}$$

After averaging over choices of x & y, winning prob − losing prob

$$p_W - p_L =$$

$$\frac{1}{4}\langle\psi|\, A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 |\psi\rangle$$

one can check that all four terms are equal to $\frac{1}{\sqrt{2}}$ $\Rightarrow$

$$p_W - p_L = \frac{1}{\sqrt{2}}$$

combined w/ $p_W + p_L = 1$, we get

$$p_W = \frac{1}{2} + \frac{1}{2\sqrt{2}} = \cos^2(\pi/8)$$

Tsirelson's bound

one cannot beat $\cos^2(\pi/8)$ in the CHSH game

Consider any observables A_0, A_1, B_0, B_1 w/ eigenvalues ~~in~~ $\{-1, 1\}$ + any state $|\psi\rangle$ such that $A_i^2 = B_j^2 = I$

then

$$\left[A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1\right]^2$$

$$= 4I - [A_0, A_1] \otimes [B_0, B_1]$$

then use that

$$\| [A_0, A_1] \|_\infty \leq 2 \|A_0\|_\infty \|A_1\|_\infty \leq 2$$

$$\Rightarrow \langle\psi| \cdots |\psi\rangle \leq \sqrt{8} = 2\sqrt{2}$$

$\Rightarrow$ bound on winning prob.