Trading resources in quantum Shannon theory

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Main message

- Question: *What are the net rates at which a sender and receiver can generate classical communication, quantum communication, and entanglement by using a channel many times?*
- Many special cases are known, such as the classical capacity theorem [Hol98, SW97], quantum capacity theorem [Sch96, SN96, BNS98, BKN00, Llo97, Sho02, Dev05], and the entanglement-assisted classical capacity theorem [BSST02]
- A priori, this question might seem challenging, but there is a surprisingly simple answer for several channels of interest:

  Just combine a single protocol with teleportation, super-dense coding, and entanglement distribution

Background — resources

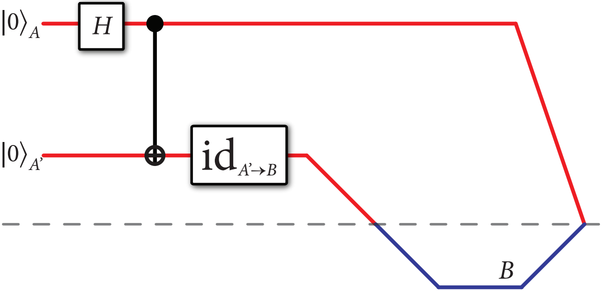
Resources [Ben04, DHW04, DHW08]

- Let $[c \to c]$ denote a noiseless classical bit channel from Alice (sender) to Bob (receiver), which performs the following mapping on a qubit density operator

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix} \to \begin{bmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{bmatrix}$$

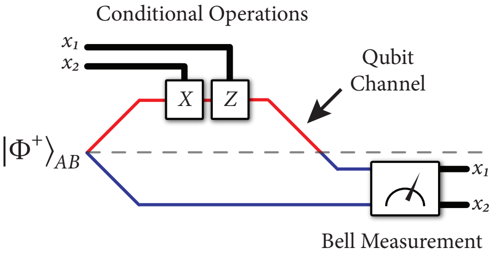
- Let $[q \to q]$ denote a noiseless quantum bit channel from Alice to Bob, which perfectly preserves a qubit density operator.
- Let $[qq]$ denote a noiseless ebit shared between Alice and Bob, which is a maximally entangled state $|\Phi^+\rangle_{AB} = (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$.
- Entanglement distribution, super-dense coding, and teleportation are non-trivial protocols for combining these resources

Entanglement distribution



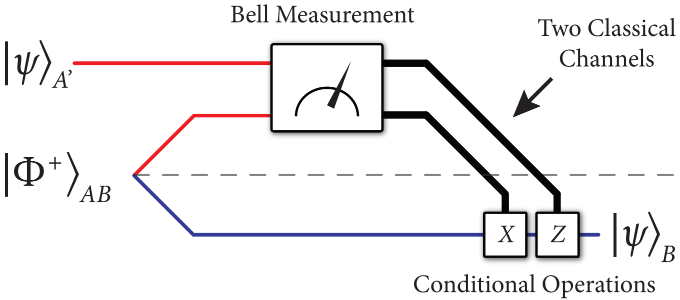
- Alice performs local operations (the Hadamard and CNOT) and consumes one use of a noiseless qubit channel to generate one noiseless ebit $|\Phi^+\rangle_{AB}$ shared with Bob.
- Resource inequality: $[q \to q] \geq [qq]$

Super-dense coding [BW92]



- Alice and Bob share an ebit. Alice would like to transmit two classical bits $x_1 x_2$ to Bob. She performs a Pauli rotation conditioned on $x_1 x_2$ and sends her share of the ebit over a noiseless qubit channel. Bob then performs a Bell measurement to get $x_1 x_2$.
- Resource inequality: $[q \to q] + [qq] \geq 2[c \to c]$

Teleportation [BBC+93]



- Alice would like to transmit an arbitrary quantum state $|\psi\rangle_{A'}$ to Bob. Alice and Bob share an ebit before the protocol begins. Alice can "teleport" her quantum state to Bob by consuming the entanglement and two uses of a noiseless classical bit channel.
- Resource inequality: $2[c \to c] + [qq] \geq [q \to q]$

Combining protocols [HW10]

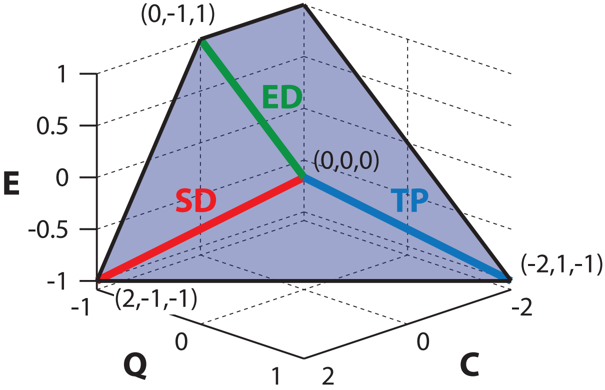
- Think of each protocol as a rate triple (C, Q, E)
- Entanglement distribution is $(0, -1, 1)$
- Super-dense coding is $(2, -1, -1)$
- Teleportation is $(-2, 1, -1)$
- All achievable rate triples are then given by

$$\{(C, Q, E) = \alpha(-2, 1, -1) + \beta(2, -1, -1) + \gamma(0, -1, 1) : \alpha, \beta, \gamma \geq 0\}$$

- Writing as a matrix equation, inverting, and applying constraints $\alpha, \beta, \gamma \geq 0$ gives the following achievable rate region:

$$\begin{aligned} C + Q + E &\leq 0, \\ Q + E &\leq 0, \\ C + 2Q &\leq 0. \end{aligned}$$

Unit resource capacity region [HW10]



The unit resource capacity region is $C + Q + E \leq 0$, $Q + E \leq 0$, $C + 2Q \leq 0$ and is provably optimal.

Trading resources using a quantum channel

- Main question: What net rates of classical communication, quantum communication, and entanglement generation can we achieve by using a quantum channel $\mathcal{N}$ many times?

- That is, what are the rates $C_{\text{out}}, Q_{\text{out}}, E_{\text{out}}, C_{\text{in}}, Q_{\text{in}}, E_{\text{in}} \geq 0$ achievable in the following resource inequality?

$$\langle \mathcal{N} \rangle + C_{\text{in}}[c \to c] + Q_{\text{in}}[q \to q] + E_{\text{in}}[qq] \\ \geq C_{\text{out}}[c \to c] + Q_{\text{out}}[q \to q] + E_{\text{out}}[qq]$$

- The union of all achievable rate triples $(C_{\text{out}} - C_{\text{in}}, Q_{\text{out}} - Q_{\text{in}}, E_{\text{out}} - E_{\text{in}})$ is called the quantum dynamic capacity region.

Trading resources using a quantum channel

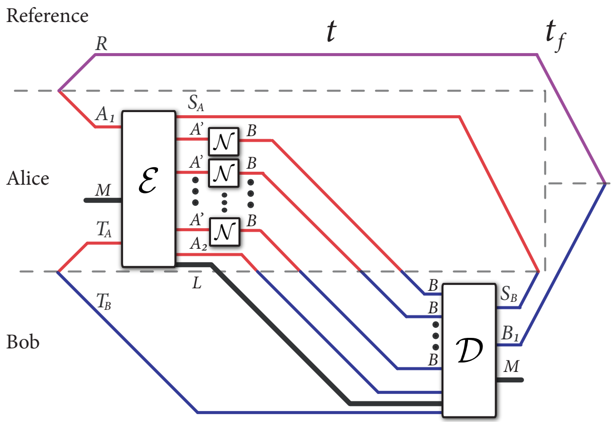


Figure: The most general protocol for generating classical communication, quantum communication, and entanglement with the help of the same respective resources and many uses of a quantum channel.

Background — entropies

- The optimal rates are expressed in terms of entropies, which we review briefly
- Given a density operator σ, the quantum entropy is defined as $H(\sigma) = -\operatorname{Tr}\{\sigma \log \sigma\}$.
- Given a bipartite density operator ρ_{AB}, the quantum mutual information is defined as

$$I(A;B)_\rho = H(A)_\rho + H(B)_\rho - H(AB)_\rho$$

- The coherent information $I(A\rangle B)_\rho$ is defined as

$$I(A\rangle B)_\rho = H(B)_\rho - H(AB)_\rho$$

- Given a tripartite density operator ρ_{ABC}, the conditional mutual information is defined as

$$I(A;B|C)_\rho = H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho$$

Quantum dynamic capacity theorem (setup) [WH12]

Define the state-dependent region $\mathcal{C}^{(1)}_{\mathrm{CQE},\sigma}(\mathcal{N})$ as the set of all rates C, Q, and E, such that

$$
\begin{aligned}
C + 2Q &\leq I(AX;B)_\sigma, \\
Q + E &\leq I(A\rangle BX)_\sigma, \\
C + Q + E &\leq I(X;B)_\sigma + I(A\rangle BX)_\sigma.
\end{aligned}
$$

The above entropic quantities are with respect to a classical–quantum state σ_{XAB}, where

$$
\sigma_{XAB} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A'\to B}(\phi^x_{AA'}),
$$

and the states $\phi^x_{AA'}$ are pure.

Quantum dynamic capacity theorem (statement) [WH12]

Define $\mathcal{C}^{(1)}_{\mathsf{CQE}}(\mathcal{N})$ as the union of the state-dependent regions $\mathcal{C}^{(1)}_{\mathsf{CQE},\sigma}(\mathcal{N})$:

$$\mathcal{C}^{(1)}_{\mathsf{CQE}}(\mathcal{N}) \equiv \bigcup_{\sigma} \mathcal{C}^{(1)}_{\mathsf{CQE},\sigma}(\mathcal{N}).$$

Then the quantum dynamic capacity region $\mathcal{C}_{\mathsf{CQE}}(\mathcal{N})$ of a channel $\mathcal{N}$ is equal to the following expression:

$$\mathcal{C}_{\mathsf{CQE}}(\mathcal{N}) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} \mathcal{C}^{(1)}_{\mathsf{CQE}}(\mathcal{N}^{\otimes k})},$$

where the overbar indicates the closure of a set.

It is implicit that one should consider states on A'^k instead of A' when taking the regularization.

Example: Qubit dephasing channel

- Take the channel to be the qubit dephasing channel $\mathcal{N}(\rho) = (1-p)\rho + pZ\rho Z$ with dephasing parameter $p = 0.2$.
- Take the input state as

$$\sigma_{XAA'} \equiv \frac{1}{2}(|0\rangle\langle 0|_X \otimes \phi^0_{AA'} + |1\rangle\langle 1|_X \otimes \phi^1_{AA'}),$$

where

$$\left|\phi^0\right\rangle_{AA'} \equiv \sqrt{1/4}|00\rangle_{AA'} + \sqrt{3/4}|11\rangle_{AA'},$$
$$\left|\phi^1\right\rangle_{AA'} \equiv \sqrt{3/4}|00\rangle_{AA'} + \sqrt{1/4}|11\rangle_{AA'}.$$

- The state σ_{XAB} resulting from the channel is $\mathcal{N}_{A'\to B}(\sigma_{XAA'})$

Example: Qubit dephasing channel (ctd.)

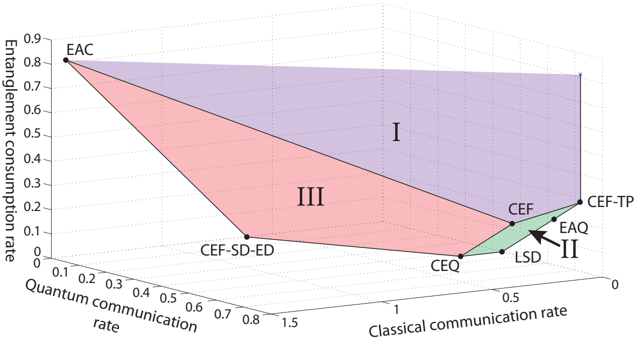


Figure: An example of the state-dependent achievable region $\mathcal{C}^{(1)}_{\text{CQE}\,\sigma}(\mathcal{N})$ corresponding to a state σ_{XABE} that arises from a qubit dephasing channel with dephasing parameter $p = 0.2$. The figure depicts the octant corresponding to the consumption of entanglement and the generation of classical and quantum communication.

Direct part of the quantum dynamic capacity theorem

Entanglement-assisted classical and quantum communication

- There is a protocol that implements the following resource inequality:

$$\langle \mathcal{N} \rangle + \frac{1}{2} I(A;E|X)_\rho \, [qq] \geq \frac{1}{2} I(A;B|X)_\rho \, [q \to q] + I(X;B)_\rho \, [c \to c]$$

where ρ_{XABE} is a state of the following form:

$$\rho_{XABE} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{U}^{\mathcal{N}}_{A' \to BE}(\varphi^x_{AA'}),$$

the states $\varphi^x_{AA'}$ are pure, and $U^{\mathcal{N}}_{A' \to BE}$ is an isometric extension of the channel $\mathcal{N}_{A' \to B}$.

- Combine this with the unit protocols of teleportation, super-dense coding, and entanglement distribution

Direct part of the quantum dynamic capacity theorem

- Combining the protocols gives the following set of achievable rates:

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} I(X;B)_\sigma \\ \frac{1}{2} I(A;B|X)_\sigma \\ -\frac{1}{2} I(A;E|X)_\sigma \end{bmatrix},$$

  where $\alpha,\ \beta,\ \gamma \geq 0$.

- Inverting the matrix equation, applying the constraints $\alpha,\ \beta,\ \gamma \geq 0$, and using entropy identities gives the following region:

$$C + 2Q \leq I(AX;B)_\sigma,$$
$$Q + E \leq I(A\rangle BX)_\sigma,$$
$$C + Q + E \leq I(X;B)_\sigma + I(A\rangle BX)_\sigma,$$

  which establishes the achievability part.

Direct part of the quantum dynamic capacity theorem

How to achieve the following resource inequality?

$$\langle \mathcal{N} \rangle + \frac{1}{2} I(A;E|X)_\rho \, [qq] \geq \frac{1}{2} I(A;B|X)_\rho \, [q \to q] + I(X;B)_\rho \, [c \to c]$$

Tools for achievability part [Wil15, Chapter 25]

- HSW classical capacity theorem [Hol98, SW97]
- Entanglement-assisted classical capacity theorem [BSST02] (see also [HDW08])
- Modification of a classical trick called "superposition coding" [Sho04]
- Another trick called coherent communication [Har04, DHW08]

HSW theorem (constant-composition variant)

- Fix an ensemble $\{p_X(x), \rho_A^x\}$ and set $\sigma_B^x \equiv \mathcal{N}_{A\to B}(\rho_A^x)$.
- Now select a typical type class T_t, which is a set of all the sequences x^n with
  1. the same empirical distribution $t(x)$
  2. $t(x)$ deviates from the distribution $p_X(x)$ by no more than $\delta > 0$
- All the sequences in the same type class are related to one another by a permutation, and all of them are strongly typical
- Select a code at random by picking all of the codewords independently and uniformly at random from the typical type class
- We can then conclude the existence of a codebook $\{x^n(m)\}_{m\in\mathcal{M}}$ and a decoding POVM $\{\Lambda_{B^n}^m\}_{m\in\mathcal{M}}$ such that $\mathcal{M} \approx 2^{nI(X;B)}$ and

$$\text{Tr}\left\{\Lambda_{B^n}^m \mathcal{N}^{\otimes n}\left(\rho_{A^n}^{x^n(m)}\right)\right\} \geq 1 - \varepsilon \quad \forall m \in \mathcal{M}$$

Entanglement-assisted coding (simple version)

- Allow Alice and Bob to share a maximally entangled state $|\Phi\rangle_{AB}$
- They then induce the following ensemble by Alice applying a Heisenberg–Weyl operator uniformly at random:

$$\left\{ d^{-2}, (\mathcal{N}_{A\to B'} \otimes \mathrm{id}_B)\,(\Phi^{x,z}_{AB}) \right\}.$$

  where $|\Phi^{x,z}\rangle_{AB} = X(x)_A\ Z(z)_A\,|\Phi\rangle_{AB}$. (This is the same ensemble from super-dense coding if $\mathcal{N}$ is the identity channel.)

- By the HSW theorem and some entropy manipulations, we can conclude that the mutual information $I(B';B)_{\mathcal{N}(\Phi)}$ is an achievable rate.

Entanglement-assisted coding (general version)

- Allow Alice and Bob to share many copies of a pure bipartite state

$$|\varphi\rangle_{AB} \equiv \sum_{x} \sqrt{p_X(x)} |x\rangle_A |x\rangle_B .$$

- Much degeneracy in many copies of this state—can rewrite it as

$$|\varphi\rangle_{AB}^{\otimes n} = \sum_{x^n} \sqrt{p_{X^n}(x^n)} |x^n\rangle_{A^n} |x^n\rangle_{B^n} = \sum_{t} \sqrt{p(t)} |\Phi_t\rangle_{A^n B^n}$$

  where $|\Phi_t\rangle_{A^n B^n}$ is maximally entangled on a type class subspace t.
- Take encoding unitary to have the form

$$U(s) \equiv \bigoplus_{t} (-1)^{b_t} \, V(x_t, z_t)$$

  where $V(x_t, z_t)$ is a Heisenberg–Weyl operator for a type class subspace t and $s = ((x_t, z_t, b_t)_t)$.

Entanglement-assisted coding (general version)

- Random coding: pick encoding unitaries $U(s)$ uniformly at random
- The entanglement-assisted quantum codewords

$$|\varphi_m\rangle_{A^n B^n} = (U_{A^n}(s(m)) \otimes I_{B^n}) |\varphi\rangle_{AB}^{\otimes n}$$

have the following interesting property:

$$|\varphi_m\rangle_{A^n B^n} = \left(I_{A^n} \otimes U_{B^n}^T(s(m))\right) |\varphi\rangle_{AB}^{\otimes n},$$

which allows us to conclude that the reduced state on the channel input is the same for all codewords:

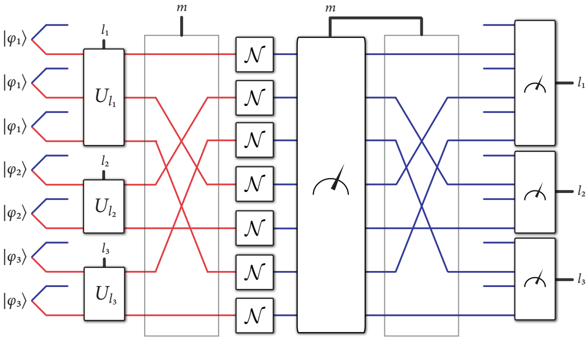
$$\mathrm{Tr}_{B^n}\{|\varphi_m\rangle\langle\varphi_m|_{A^n B^n}\} = \varphi_A^{\otimes n}$$

(privacy without access to Bob's share of the entanglement)

- Can achieve the mutual information rate $I(B'; B)_{\mathcal{N}(\varphi)}$

"Superposition coding" [Sho04]

- "Layer" an HSW code "on top of" several EA codes:



- This achieves the following resource inequality:

$$\langle \mathcal{N} \rangle + H(A|X)_\rho \, [qq] \geq I(A;B|X)_\rho \, [c \to c] + I(X;B)_\rho \, [c \to c]$$

where $\rho_{XAB} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A' \to B}(\varphi^x_{AA'})$.

Upgrading with coherent communication [Har04, DHW08]

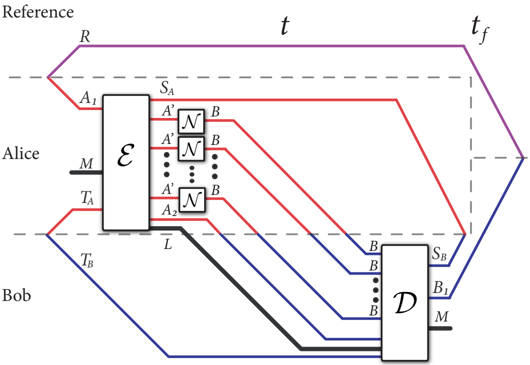
- It is possible to "upgrade" the classical bits transmitted by the entanglement-assisted codes to "coherent bits", because they are private from the environment of the channel [DHW08]
- We can then use a trick called the coherent communication identity [Har04] to conclude that the desired resource inequality is achievable:

$$\langle \mathcal{N} \rangle + \frac{1}{2} I(A;E|X)_\rho \, [qq] \geq \frac{1}{2} I(A;B|X)_\rho \, [q \to q] + I(X;B)_\rho \, [c \to c]$$

where $\rho_{XABE} \equiv \sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{U}^{\mathcal{N}}_{A' \to BE}(\varphi^x_{AA'})$.

Converse part

- Consider the most general protocol:



- Make use of quantum data processing and dimension bounds for information quantities

Computing the boundary of the region [WH12]

- Let $\vec{w} \equiv (w_C, w_Q, w_E) \in \mathbb{R}^3$ be a weight vector, $\vec{R} \equiv (C, Q, E)$ a rate vector, and $\mathcal{E} \equiv \{p_X(x), \phi^x_{AA'}\}$ an ensemble.

- Can phrase the task of computing the boundary of the single-copy capacity region as an optimization problem:

$$\begin{aligned} P^*(\vec{w}) \equiv \sup_{\vec{R},\mathcal{E}} \vec{w} \cdot \vec{R} \\ \text{subject to} \quad C + 2Q \leq I(AX;B)_\sigma, \\ Q + E \leq I(A\rangle BX)_\sigma, \\ C + Q + E \leq I(X;B)_\sigma + I(A\rangle BX)_\sigma, \end{aligned}$$

where the optimization is with respect to all rate vectors $\vec{R}$ and ensembles $\mathcal{E}$, with σ_{XAB} a state of the previously given form.

Quantum dynamic capacity formula [WH12]

- By linear programming duality, if $P^*(\vec{w}) < \infty$, then the optimization problem is equivalent to computing the quantum dynamic capacity formula, defined as

$$D_{\vec{\lambda}}(\mathcal{N}) \equiv \max_{\sigma} \lambda_1 I(AX;B)_\sigma + \lambda_2 I(A\rangle BX)_\sigma + \lambda_3 \left[I(X;B)_\sigma + I(A\rangle BX)_\sigma \right],$$

  where σ_{XAB} is a state of the previously given form and $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \lambda_3)$ is a vector of Lagrange multipliers such that $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

- Suppose for a given channel $\mathcal{N}$ that $D_{\vec{\lambda}}(\mathcal{N}^{\otimes n}) = nD_{\vec{\lambda}}(\mathcal{N}) \;\; \forall n \geq 1$ and $\vec{\lambda} \succeq 0$. Then the computation of the boundary simplifies significantly. This happens for a number of important channels.

Example: Quantum erasure channel

- Erasure channel is defined as follows:

$$\mathcal{N}^{\varepsilon}(\rho) = (1-\varepsilon)\,\rho + \varepsilon|e\rangle\langle e|,$$

  where ρ is a d-dimensional input state, $|e\rangle$ is an erasure flag state orthogonal to all inputs (so that the output space has dimension $d+1$), and $\varepsilon \in [0,1]$ is the erasure probability.

- Let $\mathcal{N}^{\varepsilon}$ be a quantum erasure channel with $\varepsilon \in [0,1/2]$. Then the quantum dynamic capacity region $\mathcal{C}_{\mathrm{CQE}}(\mathcal{N}^{\varepsilon})$ is equal to the union of the following regions, obtained by varying $\lambda \in [0,1]$:

$$\begin{aligned} C + 2Q &\leq (1-\varepsilon)\,(1+\lambda)\log d, \\ Q + E &\leq (1-2\varepsilon)\,\lambda \log d, \\ C + Q + E &\leq (1-\varepsilon-\varepsilon\lambda)\log d. \end{aligned}$$

Example: Quantum erasure channel

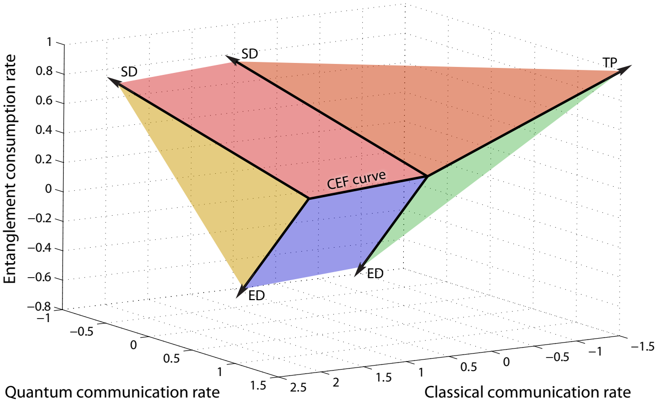


Figure: The quantum dynamic capacity region for the (qubit) quantum erasure channel with $\varepsilon = 1/4$. The plot demonstrates that time-sharing is optimal.

Example: Qubit dephasing channel

The dynamic capacity region $\mathcal{C}_{\text{CQE}}(\overline{\Delta}_p)$ of a dephasing channel with dephasing parameter $p \in [0,1]$ is the set of all C, Q, and E such that

$$
\begin{aligned}
C + 2Q &\leq 1 + h_2(\nu) - h_2(\gamma(\nu,p)), \\
Q + E &\leq h_2(\nu) - h_2(\gamma(\nu,p)), \\
C + Q + E &\leq 1 - h_2(\gamma(\nu,p)),
\end{aligned}
$$

where $\nu \in [0, 1/2]$, h_2 is the binary entropy function, and

$$
\gamma(\nu,p) \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 16 \cdot \frac{p}{2}\left(1 - \frac{p}{2}\right)\nu(1-\nu)}.
$$

Example: Qubit dephasing channel

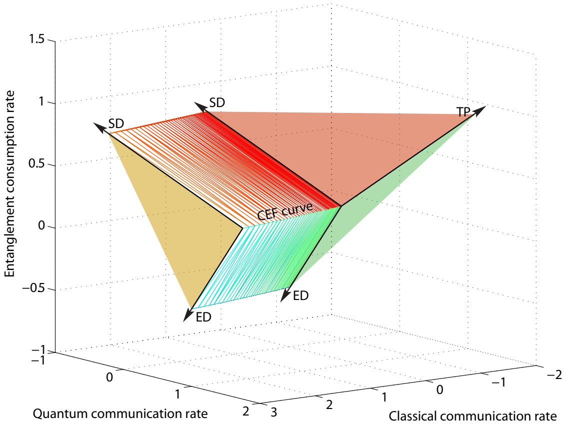


Figure: A plot of the dynamic capacity region for a qubit dephasing channel with dephasing parameter $p = 0.2$. Slight improvement over time-sharing.

Example: Pure-loss bosonic channel

- Pure-loss channel is defined from the following input-output relation:

$$\begin{aligned} \hat{a} &\rightarrow \hat{b} = \sqrt{\eta}\, \hat{a} + \sqrt{1-\eta}\, \hat{e}, \\ \hat{e} &\rightarrow \hat{e}' = -\sqrt{1-\eta}\, \hat{a} + \sqrt{\eta}\, \hat{e}, \end{aligned}$$

  where $\hat{a}$ is the input annihilation operator for the sender, $\hat{e}$ is the input annihilation operator for the environment, and $\eta \in [0,1]$ is the transmissivity of the channel.

- Place a photon number constraint on the input mode to the channel, such that the mean number of photons at the input cannot be greater than $N_S \in [0, \infty)$.

Example: Pure-loss bosonic channel [WHG12]

Build trade-off codes from an ensemble of the following form:

$$\left\{p_{(1-\lambda)N_S}(\alpha), D_{A'}(\alpha)|\psi_{\text{TMS}}(\lambda)\rangle_{AA'}\right\},$$

where $\alpha \in \mathbb{C}$,

$$p_{(1-\lambda)N_S}(\alpha) \equiv \frac{1}{\pi(1-\lambda)N_S}\exp\left\{-|\alpha|^2/[(1-\lambda)N_S]\right\},$$

$\lambda \in [0,1]$ is a photon-number-sharing parameter, $D_{A'}(\alpha)$ is a "displacement" unitary operator acting on system A', and $|\psi_{\text{TMS}}(\lambda)\rangle_{AA'}$ is a "two-mode squeezed" (TMS) state:

$$|\psi_{\text{TMS}}(\lambda)\rangle_{AA'} \equiv \sum_{n=0}^{\infty}\sqrt{\frac{[\lambda N_S]^n}{[\lambda N_S+1]^{n+1}}}|n\rangle_A|n\rangle_{A'},$$

Example: Pure-loss bosonic channel [WHG12]

The quantum dynamic capacity region for a pure-loss bosonic channel with transmissivity $\eta \geq 1/2$ is the union of regions of the form:

$$\begin{aligned} C + 2Q &\leq g(\lambda N_S) + g(\eta N_S) - g((1-\eta)\,\lambda N_S), \\ Q + E &\leq g(\eta\lambda N_S) - g((1-\eta)\,\lambda N_S), \\ C + Q + E &\leq g(\eta N_S) - g((1-\eta)\,\lambda N_S), \end{aligned}$$

where $\lambda \in [0,1]$ is a photon-number-sharing parameter and $g(N)$ is the entropy of a thermal state with mean photon number N. (This holds provided that an unsolved minimum-output entropy conjecture is true.) The region is still achievable if $\eta < 1/2$.

Example: Pure-loss bosonic channel [WHG12]

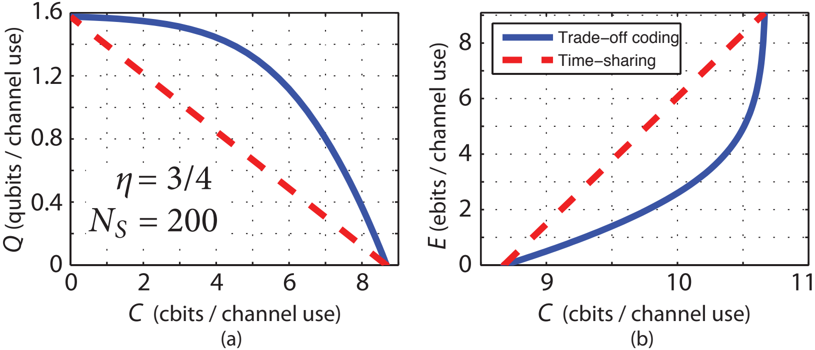


Figure: Suppose channel transmits on average 3/4 of the photons to the receiver, while losing the other 1/4 en route. Take mean photon budget of about 200 photons per channel use at the transmitter. (a) classical–quantum trade-off, (b) classical comm. with rate-limited entanglement consumption. Big gains over time-sharing.

Conclusion

Summary

- The quantum dynamic capacity theorem characterizes the net rates at which a sender and a receiver can generate classical communication, quantum communication, and entanglement by using a quantum channel many times
- The region simplifies for several channels of interest

Open questions

- Is there a simple characterization for distillation tasks? For progress, see [HW10]
- Can we sharpen the theorem? Strong converse bounds, error exponents, finite-length, second-order, etc.
- What about channel simulation tasks? (see, e.g., [BDH$^+$14])

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