Strong converse for the classical capacity of entanglement-breaking channels

1306.1586

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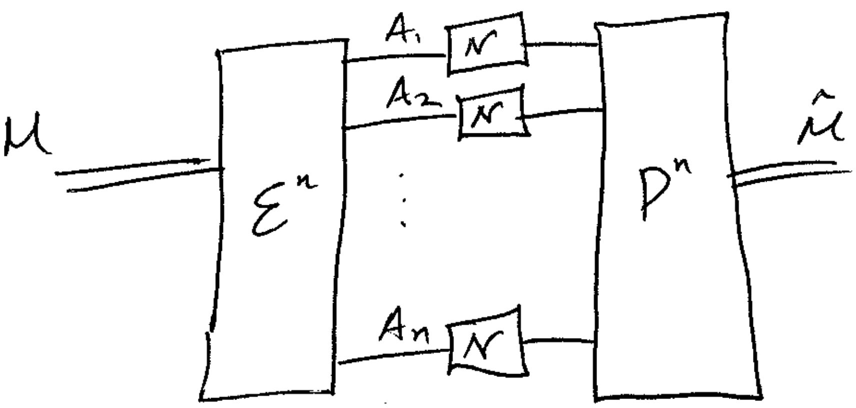
Information Processing Task:

Transmission of classical data over a quantum channel



(n, R, ε) protocol uses channel n times at rate $R = \frac{\log |M|}{n}$ + $\Pr\{\hat{M} \neq M\} \le \varepsilon$

a rate $R = \frac{\log |M|}{n}$ is achievable if $\exists$ an (n, R, ε) protocol $\forall \varepsilon > 0$ + suff. large n.

classical capacity is the supremum of all achievable rates.

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Holevo - Schumacher - Westmoreland theorem

classical capacity of any channel is equal to

$$\lim_{n\to\infty} \frac{1}{n} \chi(\mathcal{N}^{\otimes n})$$

where

$$\chi(\mathcal{N}) = \max_{\{p_X(x), \rho_x\}} I(X;B)$$

$I(X;B) = H(X) + H(B) - H(XB)$ ← mut. info. w.r.t. state

$$\sum_x p_X(x) |x\rangle\langle x|_X \otimes \mathcal{N}_{A\to B}(\rho_x)$$

proof consists of 3 parts:

1) prove that if $R \leq \chi(\mathcal{N})$ then

prob. error. $\leq 2^{-n\delta}$

for some $\delta > 0$

can then code for superchannel $\mathcal{N}^{\otimes k}$ to achieve ~~the~~ rate $\frac{1}{k}\chi(\mathcal{N}^{\otimes k})$

2) weak converse - any (n, R, ε) protocol satisfies

$$R \leq \frac{1}{n(1-\varepsilon)} \left[\chi(\mathcal{N}^{\otimes n}) + h_2(\varepsilon)\right]$$

3) try to show additivity of χ

weak converse ~~[illegible]~~ leaves room for a trade-off between rate & error, i.e., by increasing $\varepsilon > 0$, can we achieve a higher rate of communication?

strong converse shows that this is not possible. That is, if $R > C(\mathcal{N})$, then

prob. error $\geq 1 - 2^{-n\delta}$

for some constant $\delta > 0$.

Implication: capacity is a very sharp dividing line between what is possible & impossible if a strong converse exists.

This work shows that a strong converse theorem holds for all entanglement-breaking channels (recently showed for Hadamard channels also)

Background:

Any linear map $\mathcal{M}_{A\to B}$ on the space of [redacted] operators can be written as

$$\mathcal{M}_{A\to B}(X) = \sum_x N_x \operatorname{Tr}\{M_x X\}$$

for some operators $\{M_x\}$ & $\{N_x\}$.

Such a map is EB if

$N_x, M_x \geq 0 \quad \forall x$ & it is also completely positive as well.

Also, it holds for any state ρ_{12} that

[redacted] $(\mathcal{M}_{EB} \otimes id)(\rho_{12}) =$

$$\sum_z F_z \otimes G_z$$

where $F_z, G_z \geq 0 \ \forall z$

Important observation:

Conjugating an EB map by a positive operator does not take it out of the EB class b/c

$$\mathcal{M}_{EB}(X) = Y\left(\sum_x N_x \operatorname{Tr}\{M_x X\}\right) Y$$
$$= \sum_x (Y N_x Y) \operatorname{Tr}\{M_x X\}$$

$Y N_x Y \geq 0$ if $N_x \geq 0$.

An EB map ~~is~~ is an EB <u>channel</u>

if $\sum_x M_x = I$ & each N_x is a density operator.

(prepare and measure interpretation)

also $(\mathcal{M}_{EB} \otimes \mathrm{id})(\rho_{12})$ is a separable state

(namesake for EB)

Sketch of Strong Converse Proof

Let $D(\rho\|\sigma)$ denote a "generalized divergence" that satisfies

Sharma-Warsi 1205.1712

1) Monotonicity: $D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$

for q. channel $\mathcal{N}$.

2) Invariance under tensoring w/ another q. state:

$$D(\rho\otimes\tau \| \sigma\otimes\tau) = D(\rho\|\sigma)$$

3) reduction to a classical divergence when ρ & σ commute (independent of basis of ρ & σ).

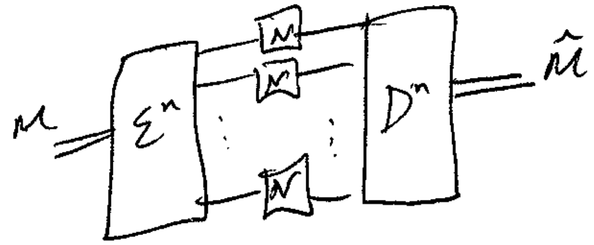
can then define a Holevo-like quantity

$$\chi_D(\mathcal{N}) = \max_{\{p(x),\rho_x\}} I_D(X;B)$$

$$I_D(X;B) = \min_{\sigma_B} D(\rho_{XB}\|\rho_X\otimes\sigma_B)$$

this satisfies data processing as well

can use this to bound the success probability of any coding scheme



$\Pr\{\hat{M} \neq M\} \leq \varepsilon$

consider cq state

$$\rho_{MB^n} \equiv \sum_m \frac{1}{M} |m\rangle\langle m|_M \otimes \mathcal{N}^{\otimes n}(\rho_m)$$

Then

$$\chi_D(\mathcal{N}^{\otimes n}) \geq I_D(M;B^n) \quad \text{(max. by def.)}$$

$$\geq I_D(M;\hat{M}) \quad \text{(data processing)}$$

$$\geq \delta(\Pr\{\hat{M} \neq M\} \,\|\, 1-2^{-nR})$$

$$\geq \delta(\varepsilon \,\|\, 1-2^{-nR})$$

monotonicity

by applying map $(M,\hat{M}) \to \delta_{M,\hat{M}}$

where $\delta(p\|q)$ is w.r.t. to dists $(p,1-p)$ + $(q,1-q)$

for von Neumann rel. ent., we get weak converse

"Sandwiched" Renyi Relative Entropy

$$\tilde{D}_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}\right\}$$

if $\operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma)$

$+\infty$ otherwise

only consider $\alpha \in (1,\infty)$

(see Müller-Lennert et al. as well 1306.3142)

Properties:

1) $\tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \operatorname{Tr}\{\rho^\alpha\sigma^{1-\alpha}\}$

2) For $\alpha \in (1,2]$,

$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

for channels $\mathcal{N}$.

3) $\lim_{\alpha\to 1}\tilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma) \equiv \operatorname{Tr}\{\rho\log\rho\} - \operatorname{Tr}\{\rho\log\sigma\}$

Monotonicity holds $\forall\, \alpha \in [1/2,\infty)$

Frank-Lieb 1306.5358

Can define a Holevo-like quantity:

$$\tilde{\chi}_\alpha(\mathcal{N}) = \max_{\{p(x),\rho_x\}} \tilde{\chi}_\alpha(\{p(x),\mathcal{N}(\rho_x)\})$$

where

$$\tilde{\chi}_\alpha(\{p(x),\rho_x\}) = \min_{\sigma_B} \tilde{D}_\alpha(\rho_{XB} \| \rho_X \otimes \sigma_B)$$

Define an "α"-information radius of channel $\mathcal{N}$ as

$$\tilde{K}_\alpha(\mathcal{N}) \equiv \min_\sigma \max_\rho \tilde{D}_\alpha(\mathcal{N}(\rho)\|\sigma)$$

<u>Theorem:</u> $\tilde{\chi}_\alpha(\mathcal{N}) = \tilde{K}_\alpha(\mathcal{N})$

Also, using definitions, we can show that

$$\tilde{K}_\alpha(\mathcal{N}) = \min_\sigma \max_\rho \frac{\alpha}{\alpha-1} \log \|\mathcal{N}(\rho)\|_{\alpha,\sigma^{\frac{1-\alpha}{\alpha}}}$$

where

$$\|A\|_{\alpha,X} \equiv \|X^{1/2} A X^{1/2}\|_\alpha$$

"sandwiched" α-norm

Using bound from before, we get

$$\widetilde{\chi}_\alpha(\mathcal{N}^{\otimes n}) \geq \widetilde{\delta}_\alpha(\varepsilon \| 1-2^{-nR}) \quad \forall \alpha \in (1,2]$$

since $\widetilde{D}_\alpha$ satisfies requirements of generalized divergence

Using $\widetilde{\delta}_\alpha(\varepsilon \| 1-2^{-nR}) =$

$$\frac{1}{\alpha-1} \log\left(\varepsilon^\alpha (1-2^{-nR})^{1-\alpha} + (1-\varepsilon)^\alpha (2^{-nR})^{1-\alpha} \right)$$

$$\geq \frac{1}{\alpha-1} \log\left((1-\varepsilon)^\alpha (2^{-nR})^{1-\alpha} \right)$$

$$= \frac{\alpha}{\alpha-1} \log(1-\varepsilon) + nR$$

can rewrite as

$$P_{succ.} \leq 2^{-n\left(\frac{\alpha-1}{\alpha}\right)\left(R - \frac{1}{n}\widetilde{\chi}_\alpha(\mathcal{N}^{\otimes n})\right)}$$

if we can show that

$$\frac{1}{n}\widetilde{\chi}_\alpha(\mathcal{N}^{\otimes n}) \leq \widetilde{\chi}_\alpha(\mathcal{N})$$

then we would be done. b/c

$$P_{succ} \leq 2^{-n\left(\frac{\alpha-1}{\alpha}\right)\left(R - \widetilde{\chi}_\alpha(\mathcal{N})\right)}$$

To show subadditivity, recall a few things:

$$r_\alpha(\mathcal{M}) \equiv \max_\rho \|\mathcal{M}(\rho)\|_\alpha$$

maximum output α-norm of CP map $\mathcal{M}$

Theorem: (King-Holevo)

$$r_\alpha(\mathcal{M}_{EB} \otimes \mathcal{M}) = r_\alpha(\mathcal{M}_{EB})\, r_\alpha(\mathcal{M})$$

From this, we get

$$\tilde{\chi}_\alpha(\mathcal{N}_{EB} \otimes \mathcal{N}) \leq \tilde{\chi}_\alpha(\mathcal{N}_{EB}) + \tilde{\chi}_\alpha(\mathcal{N})$$

for channels $\mathcal{N}_{EB}, \mathcal{N}$

b/c.

$$\tilde{\chi}_\alpha(\mathcal{N}_{EB} \otimes \mathcal{N}) = \tilde{K}_\alpha(\mathcal{N}_{EB} \otimes \mathcal{N})$$

$$= \min_{\sigma_{B_1B_2}} \frac{\alpha}{\alpha-1} \log \max_{\rho_{A_1A_2}} \|(\mathcal{N}_{EB} \otimes \mathcal{N})(\rho_{A_1A_2})\|_{\alpha, \sigma_{B_1B_2}^{(1-\alpha)/\alpha}}$$

$$\leq \min_{\sigma_{B_1} \otimes \sigma_{B_2}} \frac{\alpha}{\alpha-1} \log \max_{\rho_{A_1A_2}} \|(\mathcal{N}_{EB} \otimes \mathcal{N})(\rho_{A_1A_2})\|_{\alpha, \sigma_{B_1}^{(1-\alpha)/\alpha} \otimes \sigma_{B_2}^{(1-\alpha)/\alpha}}$$

$$\le \min_{\sigma_{B_1}\otimes\sigma_{B_2}} \frac{\alpha}{\alpha-1} \log \Big[\max_{\rho_{A_1}} \| \mathcal{N}_{EB}(\rho_{A_1}) \|_{\alpha,\sigma_{B_1}^{(1-\alpha)/\alpha}} \cdot \max_{\rho_{A_2}} \| \mathcal{N}(\rho_{A_2}) \|_{\alpha,\sigma_{B_2}^{(1-\alpha)/\alpha}} \Big]$$

$$= \widetilde{K}_\alpha(\mathcal{N}_{EB}) + \widetilde{K}_\alpha(\mathcal{N})$$

$$= \widetilde{\chi}_\alpha(\mathcal{N}_{EB}) + \widetilde{\chi}_\alpha(\mathcal{N})$$

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by induction, we get $\forall \alpha \in (1,2]$

$$\widetilde{\chi}_\alpha(\mathcal{N}_{EB}^{\otimes n}) \le n\, \widetilde{\chi}_\alpha(\mathcal{N}_{EB})$$

Can show that if $R \ge \chi(\mathcal{N}_{EB})$

$\exists\ \beta > 1$ such that

$$\frac{\alpha-1}{\alpha}\left(R - \widetilde{\chi}_\alpha(\mathcal{N}_{EB})\right) > 0 \quad \forall \alpha \in (1,\beta)$$

then

$$p_{succ} \le 2^{-n\left(\frac{\alpha-1}{\alpha}\right)\left(R - \widetilde{\chi}_\alpha(\mathcal{N}_{EB})\right)}$$