Partial decode and forward for the quantum relay channel

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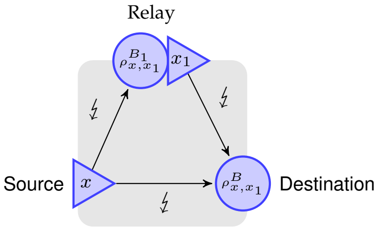
Overview

- Define the quantum relay channel
- Describe the partial decode and forward coding strategy
- Discuss briefly the proof techniques

Quantum relay channels

A classical-quantum relay channel $\mathcal{N}$ is a map with two classical inputs x and x_1 and two output quantum systems B_1 and B.



For each pair of possible input symbols $(x, x_1) \in \mathcal{X} \times \mathcal{X}_1$, the channel prepares a density operator $\rho^{B_1 B}_{x,x_1}$ defined on the tensor-product Hilbert space $\mathcal{H}^{B_1} \otimes \mathcal{H}^B$:

$$\mathcal{N}^{X X_1 \to B_1 B}(x, x_1) \equiv \rho^{B_1 B}_{x,x_1}$$

where B_1 is the Relay output and B is the Destination output.

Main idea

- The Source will transmit two kinds of messages:
  - m at a rate R_m, which will be decoded only by the Destination.
  - ℓ at a rate R_ℓ, which will be *decoded* by the Relay and *forwarded* to the Destination.
- The overall rate from Source to Destination is the sum of the two rates: $R = R_m + R_\ell$.
- The Relay is used to forwarding part of the communication (the ℓ part), while the message m is sent directly to the Destination.

Achievable rate region

Theorem (Partial decode-and-forward inner bound)

Let $\{\rho_{x,x_1}\}$ be a cc-qq relay channel. Then a rate R is achievable, provided that the following inequality holds:

$$R \leq \max_{p(u,x,x_1)} \min \left\{ \begin{array}{l} I(XX_1;B)_\theta \, , \\ I(U;B_1|X_1)_\theta + I(X;B|X_1U)_\theta \end{array} \right\} , \tag{1}$$

where the information quantities are with respect to the classical-quantum state

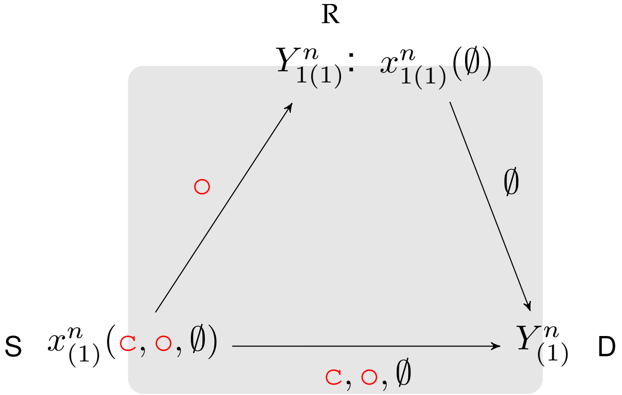
$$\theta^{UXX_1B_1B} \equiv \sum_{x,u,x_1} p(u,x,x_1) \, |u\rangle\langle u|^U \otimes |x\rangle\langle x|^X \otimes |x_1\rangle\langle x_1|^{X_1} \otimes \rho_{x,x_1}^{B_1B} .$$

Example

- The Source wants to send the string "$\texttt{co ns ti tu ti on}$" to the Destination.
- We use codewords of size n, and that each character $a \in \mathcal{ASCII}$ is treated as a message.
- The relay network will be used during seven blocks labeled by $j \in \{1, 2, 3, 4, 5, 6, 7\}$.
- The messages pairs (m_j, ℓ_j) sent during the seven uses of the channel are: $\{(\texttt{c}, \texttt{o}), (\texttt{n}, \texttt{s}), (\texttt{t}, \texttt{i}), (\texttt{t}, \texttt{u}), (\texttt{t}, \texttt{i}), (\texttt{o}, \texttt{n}), (\emptyset, \emptyset)\}$
- The Source codebook depends on the current message pair (m_j, ℓ_j) as well as the message ℓ_{j-1} of the previous block, so the transmitted codewords during the seven blocks are: $\{x^n_{(1)}(\texttt{c}, \texttt{o}, \emptyset), x^n_{(2)}(\texttt{n}, \texttt{s}, \texttt{o}), x^n_{(3)}(\texttt{t}, \texttt{i}, \texttt{s}), x^n_{(4)}(\texttt{t}, \texttt{u}, \texttt{i}), x^n_{(5)}(\texttt{t}, \texttt{i}, \texttt{u}), x^n_{(6)}(\texttt{o}, \texttt{n}, \texttt{i}), x^n_{(7)}(\emptyset, \emptyset, \texttt{n})\}$
- The Relay re-transmits the message ℓ_j during block $j + 1$: $\{x^n_{1(1)}(\emptyset), x^n_{1(2)}(\texttt{o}), x^n_{1(3)}(\texttt{s}), x^n_{1(4)}(\texttt{i}), x^n_{1(5)}(\texttt{u}), x^n_{1(6)}(\texttt{i}), x^n_{1(7)}(\texttt{n})\}$.

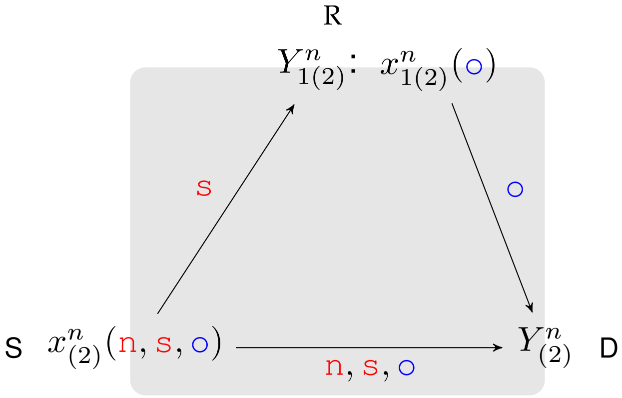
Transmission

Block 1: co ns ti tu ti on



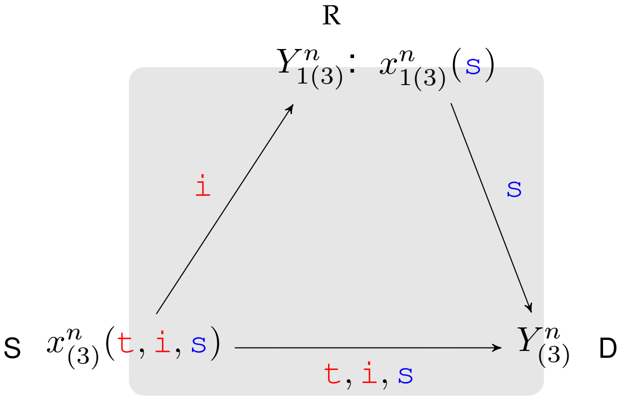
Transmission

Block 2: co ns ti tu ti on



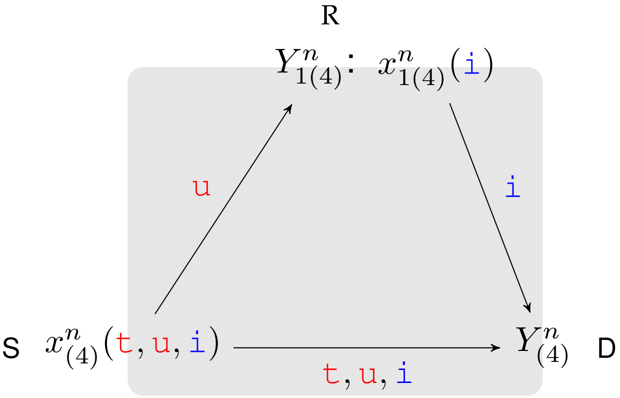
Transmission

Block 3: co ns ti tu ti on



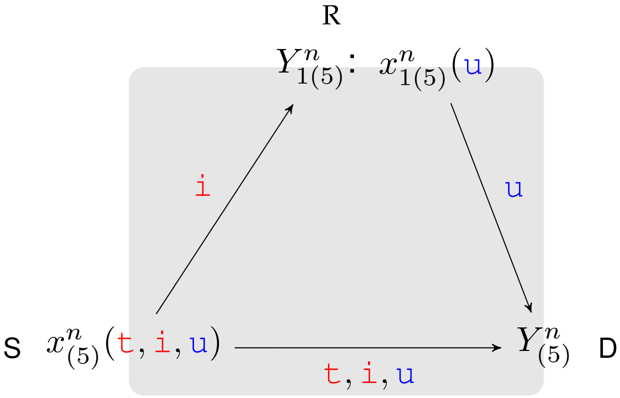
Transmission

Block 4: `co ns ti tu ti on`



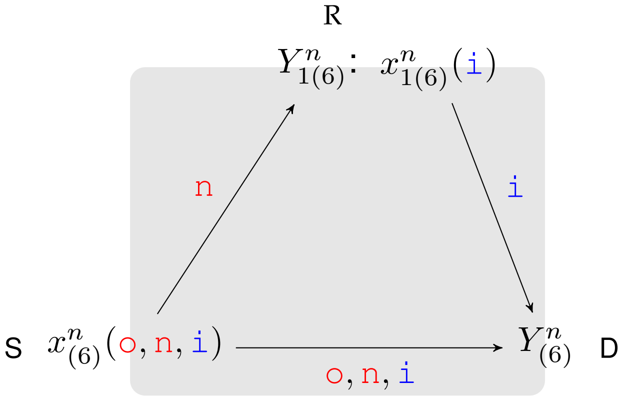
Transmission

Block 5: `co ns ti tu ti on`



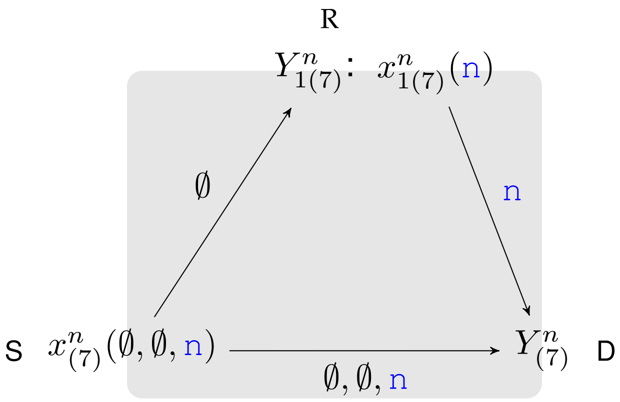
Transmission

Block 6: co ns ti tu ti on



Transmission

Block 7: `co ns ti tu ti on`



Coherent codebook construction

Fix a code distribution $p(u,x,x_1) = p(x_1)p(u|x_1)p(x|x_1,u)$.
We generate independent codebooks for each block j as follows:

- Randomly and independently generate 2^{nR_ℓ} sequences $x_1^n(\ell_{j-1})$, $\ell_{j-1} \in \left[1:2^{nR_\ell}\right]$, according to $\prod_{i=1}^{n} p(x_{1i})$.
- For each $x_1^n(\ell_{j-1})$, randomly and independently generate 2^{nR_ℓ} sequences $u^n(\ell_j,\ell_{j-1})$, $\ell_j \in \left[1:2^{nR_\ell}\right]$ according to $\prod_{i=1}^{n} p\left(u_i|x_{1i}(\ell_{j-1})\right)$.
- For each $x_1^n(\ell_{j-1})$ and each corresponding $u^n(\ell_j,\ell_{j-1})$, randomly and independently generate 2^{nR_m} sequences $x^n(m_j,\ell_j,\ell_{j-1})$, $m_j \in \left[1:2^{nR_m}\right]$, according to the distribution: $\prod_{i=1}^{n} p\Big(x_i|x_{1i}(\ell_{j-1})\,,u_i(\ell_j,\ell_{j-1})\Big)$.

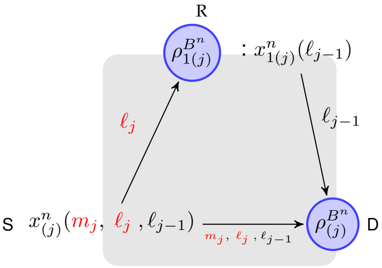
Quantum partial decode and forward strategy

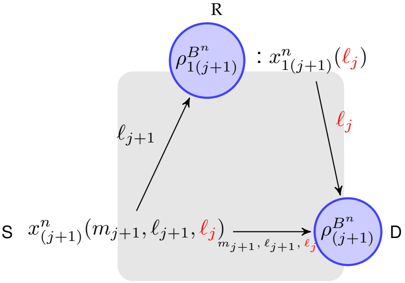
Coding strategy

- The channel is used for b blocks, indexed by $j \in \{1, \ldots, b\}$.
- We will use the coherent codebook generated according to $p(x_1)p(u|x_1)p(x|x_1,u)$ as described above.
- The Relay will decode the message ℓ_j during block j using the standard point-to-point decoding procedure.
- The Destination can simultaneously decode (m_j, ℓ_j) from a collective measurement on the output systems of blocks j and $j+1$, using a "sliding-window" measurement.

Transmissions during two consecutive blocks

The transmission of (m_j, ℓ_j) is during blocks j and $j+1$.





During block j the Sender inputs the symbol $x^n(m_j, \ell_j, \ell_{j-1})$.

During block $j+1$ the Sender inputs the symbol $x^n(m_{j+1}, \ell_{j+1}, \ell_j)$, and the Relay sends $x^n_1(\ell_j)$.

Decoding at the Destination

- In order to use the (m_j, ℓ_j) information from both the Source and the Relay, the Destination will measure the outputs of block j and $j+1$ simultaneously:

$$\rho^{(j)}_{m_j,\ell_j,\ell_{j-1}} \otimes \rho^{(j+1)}_{m_{j+1},\ell_{j+1},\ell_j}.$$

- The "sliding-window" measurement is constructed as follows:

$$P^{B^n_{(j)} B^n_{(j+1)}}_{m_j,\ell_j|\ell_{j-1}} = P^{B^n_{(j)}}_{m_j,\ell_j|\ell_{j-1}} \otimes P^{B^n_{(j+1)}}_{\ell_j},$$

where $P^{B^n_{(j)}}_{m_j,\ell_j|\ell_{j-1}} \equiv \Pi^{(j)}_{\bar{\bar{\rho}}|\ell_{j-1}} \Pi^{(j)}_{\bar{\rho}_{\ell_j|\ell_{j-1}}} \Pi^{(j)}_{\rho_{m_j,\ell_j|\ell_{j-1}}} \Pi^{(j)}_{\bar{\rho}_{\ell_j|\ell_{j-1}}} \Pi^{(j)}_{\bar{\bar{\rho}}|\ell_{j-1}}$ and $P^{B^n_{(j+1)}}_{\ell_j} \equiv \Pi^{(j+1)}_{\bar{\tau}} \Pi^{(j+1)}_{\tau_{\ell_j}} \Pi^{(j+1)}_{\bar{\tau}}$ are conditionally typical for the appropriate states.

- The Destination's measurement $\Lambda^{B^n_{(j)} B^n_{(j+1)}}_{m_j,\ell_j|\ell_{j-1}}$ is constructed from $P^{B^n_{(j)} B^n_{(j+1)}}_{m_j,\ell_j|\ell_{j-1}}$ by using the square-root normalization.

Error analysis at the Destination (1)

We consider the error analysis for a single message pair (m_j, ℓ_j):

$$\bar{p}_e^D \equiv \text{Tr}\Big[\Big(I - \Lambda^{B^n_{(j)} B^n_{(j+1)}}_{m_j, \ell_j | \ell_{j-1}}\Big) \rho^{(j)}_{m_j \ell_j \ell_{j-1}} \otimes \rho^{(j+1)}_{m_{j+1} \ell_{j+1} \ell_j}\Big]$$

$$\leq \underbrace{\ldots}_{(\text{A})} + \underbrace{\sum_{l'_j \neq l_j,\, m'_j} \text{Tr}\Big[P^{B^n_{(j)} B^n_{(j+1)}}_{m'_j, \ell'_j | \ell_{j-1}} \ \rho^{(j)}_{m_j \ell_j \ell_{j-1}} \otimes \rho^{(j+1)}_{m_{j+1} \ell_{j+1} \ell_j}\Big]}_{(\text{B})}$$

We want to calculate the expectation of average probability of error with respect to the code randomness $\mathbb{E}_{U^n X^n X_1^n}$.
We focus on the term (B):

$$\mathop{\mathbb{E}}_{U^n X^n X_1^n} \{(\text{B})\} = \mathop{\mathbb{E}}_{U^n X^n X_1^n} \sum_{\ell'_j \neq \ell_j, m'_j} \text{Tr}\Big[P^{B^n_{(j)} B^n_{(j+1)}}_{m'_j, \ell'_j | \ell_{j-1}} \ \rho^{(j)}_{m_j \ell_j \ell_{j-1}} \otimes \rho^{(j+1)}_{m_{j+1} \ell_{j+1} \ell_j}\Big]$$

Error analysis at the Destination (2)

$$= \mathop{\mathbb{E}}_{U^nX^nX_1^n} \sum_{\ell'_j \neq \ell_j, m'_j} \mathrm{Tr}\left[\left(P^{B^n_{(j)}}_{m'_j,\ell'_j|\ell_{j-1}} \otimes P^{B^n_{(j+1)}}_{\ell'_j}\right) \rho^{(j)}_{m_j\ell_j\ell_{j-1}} \otimes \rho^{(j+1)}_{m_{j+1}\ell_{j+1}\ell_j}\right]$$

$$= \mathop{\mathbb{E}}_{U^nX^nX_1^n} \sum_{\ell'_j \neq \ell_j, m'_j} \underbrace{\mathrm{Tr}\left[P^{B^n_{(j)}}_{m'_j,\ell'_j|\ell_{j-1}} \rho^{(j)}_{m_j,\ell_j,\ell_{j-1}}\right]}_{(\mathrm{B1})} \underbrace{\mathrm{Tr}\left[P^{B^n_{(j+1)}}_{\ell'_j} \rho^{(j+1)}_{m_{j+1},\ell_{j+1},\ell_j}\right]}_{(\mathrm{B2})}$$

$$= \mathop{\mathbb{E}}_{U^nX^nX_1^n} \sum_{\ell'_j \neq \ell_j, m'_j} (\mathrm{B1}) \times (\mathrm{B2})$$

$$= \sum_{\ell'_j \neq \ell_j, m'_j} \mathop{\mathbb{E}}_{U^nX^nX_1^n} \{(\mathrm{B1})\} \times \mathop{\mathbb{E}}_{U^nX^nX_1^n} \{(\mathrm{B2})\}$$

$$\leq \sum_{\ell'_j \neq \ell_j,\ m'_j} 2^{-n[I(UX;B|X_1)-2\delta]} \times 2^{-n[I(X_1;B)-2\delta]}$$

$$\leq |\mathcal{L}||\mathcal{M}|\ 2^{-n[I(X_1;B)+I(UX;B|X_1)-4\delta]}.$$

Thus, the probability of error will vanish if we choose:

$$R = R_\ell + R_m \leq I\left(X_1;B\right) + I\left(UX;B|X_1\right) - 5\delta.$$

Further reading

1. The paper [arXiv:1201.0011].
2. My thesis titled “Network information theory for classical-quantum channels”, which is a comprehensive survey of several results on P-to-P, QMAC, QIC, QBC, QRC [arXiv:1207.????].

The end

Thank you for your attention!