Classical codes for quantum broadcast channels

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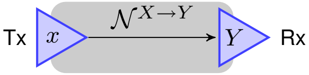
Overview

- Introduce basic notions of quantum information theory
- Define quantum broadcast channels
- Describe two different coding strategies
    1. Superposition coding inner bound
    2. Marton coding inner bound
- Discuss briefly the error analysis for the superposition coding

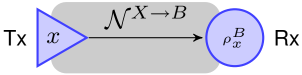
Classical-quantum channels

- A classical channel $(\mathcal{X}, \mathcal{N}^{X\to Y}(x) \equiv p_{Y|X}(y|x),\ \mathcal{Y})$



is modeled by a conditional probability distribution $p_{Y|X}$.

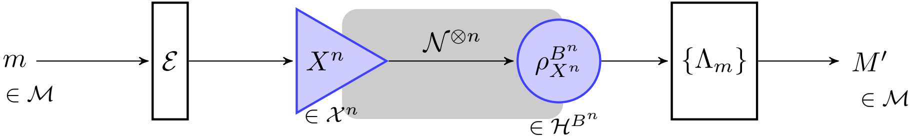
- A *classical-quantum* channel: $(\mathcal{X}, \mathcal{N}^{X\to B}(x) \equiv \rho_x^B,\ \mathcal{H}^B)$:



The channel is fully specified by the finite set of output states $\{\rho_x^B\}$ it produces for each of the possible inputs $x \in \mathcal{X}$.

Classical-quantum channel coding

We measure the number of messages $\mathcal{M} \equiv \{1, 2, \ldots, \lfloor 2^{nR} \rfloor\}$ that can be transmitted using n uses of the channel.



A rate R is *achievable* if there exists a coding scheme with small average probability of error:

$$\bar{p}_e \equiv \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \Pr\{M' \neq m \mid m \text{ is sent}\} \leq \epsilon.$$

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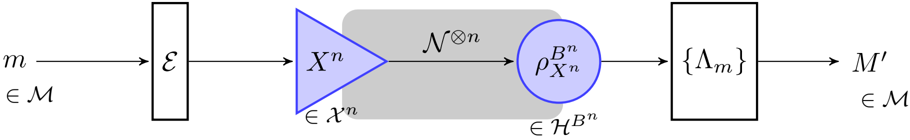

$$\bar{p}_e \equiv \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \mathrm{Tr}\left\{ \left(I - \Lambda_{x^n(m)}^{B^n} \right) \rho_{x^n(m)}^{B^n} \right\} \leq \epsilon.$$

Classical-quantum channel coding

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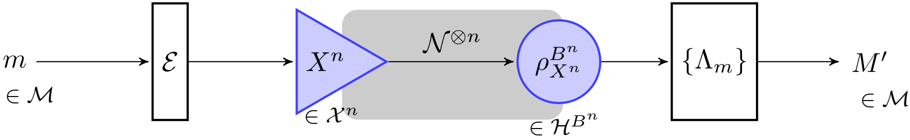


Convert noisy resource $\mathcal{N}^{X \to B}$ into noiseless resource $[c \to c]$:

$$n \cdot \mathcal{N}^{X \to B} \quad \xrightarrow{(1-\epsilon)} \quad nR \cdot [c \to c]$$

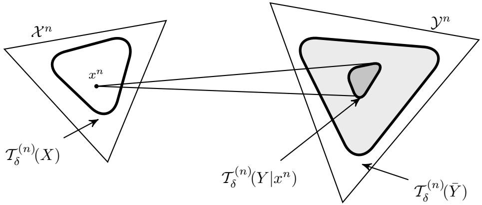
A bridge from classical to quantum information theory

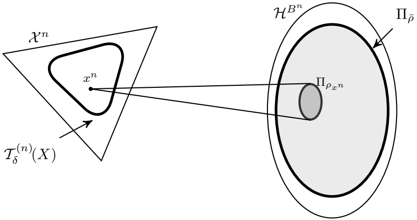
$y_a \in \mathcal{Y}$ symbol from a finite set	$\iff$	$\|v\rangle^B \in \mathcal{H}^B$ vector in a Hilbert space
$p_Y \in \mathcal{P}(\mathcal{Y})$ probability distribution $p_Y(y) \geq 0,\ \forall y \in \mathcal{Y}$ $\sum_y p_Y(y) = 1$	$\iff$	$\rho^B \in \mathcal{D}(\mathcal{H}^B)$ density matrix $\equiv$ quantum state $\langle v\|\rho^B\|v\rangle \geq 0,\ \forall\|v\rangle \in \mathcal{H}^B$ Tr$[\rho^B] = 1$
$p_{Y\|X}$ conditional probability distribution $\equiv$ classical-classical channel	$\iff$	$\{\rho_x^B\}, x \in \mathcal{X}$ conditional states $\equiv$ classical-quantum channel
$p_{XY}(x,y) \equiv p_X(x)p_{Y\|X}(y\|x)$ joint input-output distribution	$\iff$	$\theta^{XB} \equiv \sum_x p_X(x)\ \|x\rangle\langle x\|^X \otimes \rho_x^B$ joint input-output state
$p_{\bar{Y}} \equiv \mathbb{E}_X\, p_{Y\|X} = \sum_x p_{XY}(x,y)$ average output distribution	$\iff$	$\bar{\rho}^B \equiv \mathbb{E}_X\, \rho_X^B = \sum_x p_X(x)\rho_x^B$ average output state

A bridge from classical to quantum information theory

$y_a \in \mathcal{Y}$	$\Longleftrightarrow$	$\|v\rangle^B \in \mathcal{H}^B$
$p_Y \in \mathcal{P}(\mathcal{Y})$	$\Longleftrightarrow$	$\rho^B \in \mathcal{D}(\mathcal{H}^B)$
$p_{Y\|X}$	$\Longleftrightarrow$	$\{\rho_x^B\}, x \in \mathcal{X}$
$p_{XY}(x,y) \equiv p_X(x) p_{Y\|X}(y\|x)$	$\Longleftrightarrow$	$\theta^{XB} \equiv \sum_x p_X(x) \|x\rangle\langle x\|^X \otimes \rho_x^B$
$p_{\bar{Y}} \equiv \mathbb{E}_X\, p_{Y\|X} = \sum_x p_{XY}(x,y)$	$\Longleftrightarrow$	$\bar{\rho}^B \equiv \mathbb{E}_X\, \rho_X^B = \sum_x p_X(x) \rho_x^B$
$\mathbf{1}_{\left\{y^n \in \mathcal{T}_\delta^{(n)}(\bar{Y})\right\}}$ indicator function for the output-typical set	$\Longleftrightarrow$	$\Pi_{\bar{\rho}} \equiv \Pi^{B^n}_{\bar{\rho}^{\otimes n},\delta}$ projector onto the output-typical subspace
$\mathbf{1}_{\left\{y^n \in \mathcal{T}_\delta^{(n)}(Y\|x^n)\right\}}$ indicator function for the conditionally typical set	$\Longleftrightarrow$	$\Pi_{x^n} \equiv \Pi^{B^n}_{\rho_{x^n},\delta}$ conditionally typical projector for the state $\rho_{x^n}^{B^n}$

Conditional typicality





$$\mathop{\mathbb{E}}_{X^n} \sum_{y^n \in \mathcal{T}_\delta^{(n)}(Y|X^n)} p_{Y^n|X^n}\left(y^n|X^n\right) \geq 1-\epsilon,$$

$$\mathop{\mathbb{E}}_{X^n} \sum_{y^n \in \mathcal{T}_\delta^{(n)}(\bar{Y})} p_{Y^n|X^n}\left(y^n|X^n\right) \geq 1-\epsilon,$$

$$p_{\bar{Y}^n}(y^n)\, \mathbf{1}_{\left\{y^n \in \mathcal{T}_\epsilon^{(n)}(\bar{Y})\right\}} \leq 2^{-n[H(\bar{Y})-\delta]}\, \mathbf{1}_{\left\{y^n \in \mathcal{T}_\epsilon^{(n)}(\bar{Y})\right\}}$$

$$\sum_{y^n \in \mathcal{Y}^n} \mathbf{1}_{\left\{y^n \in \mathcal{T}_\epsilon^{(n)}(Y|X^n)\right\}} \leq 2^{n[H(Y|X)+\delta]}$$

$$\mathop{\mathbb{E}}_{X^n} \mathrm{Tr}\left[\rho^B_{X^n}\, \Pi^n_{\rho^B_{X^n},\delta}\right] \geq 1-\epsilon,$$

$$\mathop{\mathbb{E}}_{X^n} \mathrm{Tr}\left[\rho^B_{X^n}\, \Pi_{\bar{\rho}}\right] \geq 1-\epsilon,$$

$$\Pi_{\bar{\rho}}\bar{\rho}^{\otimes n}\, \Pi_{\bar{\rho}} \leq 2^{-n[H(B)-\delta]}\Pi_{\bar{\rho}},$$

$$\mathrm{Tr}\left[\Pi^n_{\rho^B_{X^n},\delta}\right] \leq 2^{n[H(B|X)_\rho+\delta]}.$$

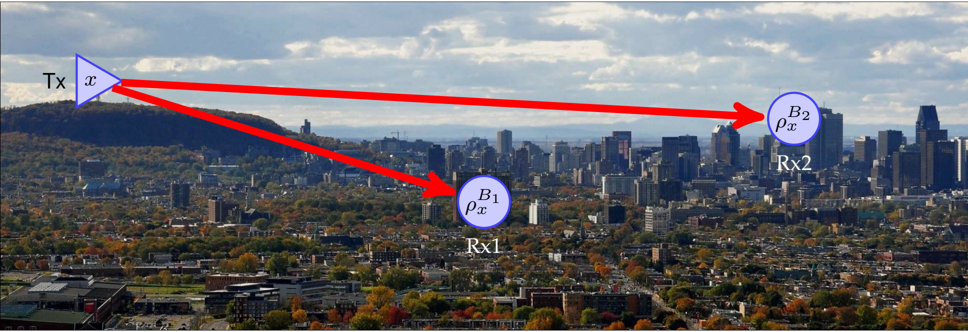
Quantum decoder construction

- How do we combine the projectors $\Pi_{\bar{\rho}}$ and $\Pi_{x^n(m)}$ to build a decoding measurement Λ_m ?

- Unlike in the classical case, the order in which we apply the typical projectors to the state matters:

$$\Pi_{x^n(m)}\Pi_{\bar{\rho}} \neq \Pi_{\bar{\rho}}\Pi_{x^n(m)}$$

Quantum broadcast channel

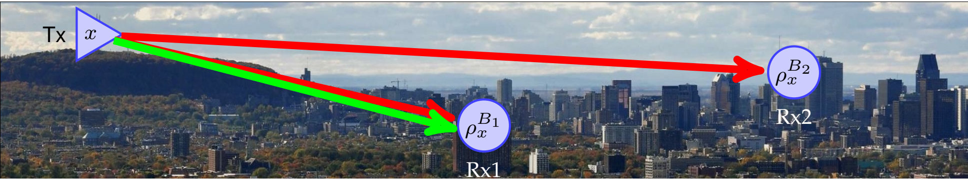


A classical-quantum broadcast channel is a triple:

$$(\mathcal{X}, \mathcal{N}^{X \to B_1 B_2} \equiv \rho_x^{B_1 B_2}, \mathcal{H}^{B_1 B_2}).$$

- The Transmitter can send one of the $|\mathcal{X}|$ possible signal states.
- Receiver 1 gets $\rho_x^{B_1} = \text{Tr}_{B_2}[\rho_x^{B_1 B_2}]$.
- Receiver 2 gets $\rho_x^{B_2} = \text{Tr}_{B_1}[\rho_x^{B_1 B_2}]$.

Superposition coding inner bound



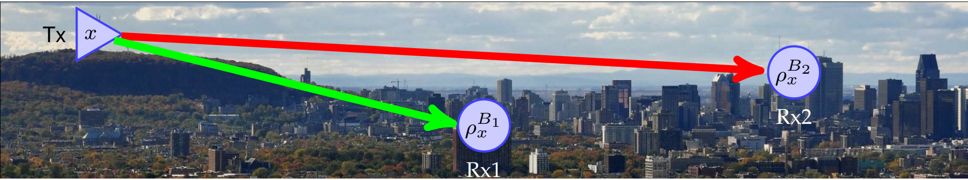
$$n \cdot \mathcal{N}^{X \to B_1 B_2} \overset{(1-\epsilon)}{\longrightarrow} nR \cdot \underbrace{[c \to cc]^{X \to B_1 B_2}}_{\text{common message}} + nR_1 \cdot \underbrace{[c \to c]^{X \to B_1}}_{\text{superimposed message}}$$

Theorem 1: A rate pair (R, R_1) is achievable for the quantum broadcast channel $\{\rho_x^{B_1 B_2}\}$ if it satisfies the following inequalities:

$$\begin{aligned} R_1 &\leq I(X; B_1 | W)_\theta, \\ R_1 + R &\leq I(X; B_1)_\theta, \\ R &\leq I(W; B_2)_\theta, \end{aligned}$$

where the above information quantities are with respect to a state $\theta^{WXB_1B_2} = \sum_{w,x} p_W(w) p_{X|W}(x|w)\, |w\rangle\langle w|^W \otimes |x\rangle\langle x|^X \otimes \rho_x^{B_1 B_2}$.

Marton inner bound



$$n \cdot \mathcal{N}^{X \to B_1 B_2} \xrightarrow{(1-\epsilon)} nR_1 \cdot \underbrace{[c \to c]^{X \to B_1}}_{\text{for Rx1}} + nR_2 \cdot \underbrace{[c \to c]^{X \to B_2}}_{\text{for Rx2}}$$

Theorem 2: Let $\{\rho_x^{B_1 B_2}\}$ be a classical-quantum broadcast channel and let $x = f(u_1, u_2)$ be a deterministic function. The following rate region is achievable:

$$\begin{aligned} R_1 &\leq I(U_1; B_1)_\theta, \\ R_2 &\leq I(U_2; B_2)_\theta, \\ R_1 + R_2 &\leq I(U_1; B_1)_\theta + I(U_2; B_2)_\theta - I(U_1; U_2)_\theta, \end{aligned} \qquad (1)$$

where the information quantities are with respect to the state:

$$\theta^{U_1 U_2 B_1 B_2} = \sum_{u_1, u_2} p(u_1, u_2) |u_1\rangle\langle u_1|^{U_1} \otimes |u_2\rangle\langle u_2|^{U_2} \otimes \rho_{f(u_1, u_2)}^{B_1 B_2}.$$

Sketch of proof of Theorem 1: Superposition coding

- The common message $\ell \in [1, \ldots, 2^{nR}]$ is encoded using a random codebook $w^n(\ell) \sim p_{W^n}(w^n)$.
- An extra message for Receiver 1, $m \in [1, \ldots, 2^{nR_1}]$ is encoded using a conditional-codebook $x^n(m, \ell) \sim p_{X^n|W^n}(x^n|w^n(\ell))$.
- Receiver 1 uses a POVM $\{\Lambda_{m,\ell}\}$ constructed from the “projector sandwich”:

$$P_{m,\ell} \;\equiv\; \Pi_{\bar{\rho}} \, \Pi_{w^n(\ell)} \, \Pi_{x^n(m,\ell)} \, \Pi_{w^n(\ell)} \, \Pi_{\bar{\rho}}, \qquad (2)$$

and the normalization procedure:

$$\Lambda_{m,\ell} \equiv \left(\sum_{m',\ell'} P_{m',\ell'} \right)^{-1/2} P_{m,\ell} \left(\sum_{m',\ell'} P_{m',\ell'} \right)^{-1/2}. \qquad (3)$$

This is known as the square-root measurement.

Sketch of proof (continued)

Without loss of generality, assume $m = 1$ and $\ell = 1$.

By using the Hayashi-Nagaoka operator inequality, we can brake up the the probability of error for Receiver 1 into two terms:

$$\bar{p}_e \equiv \text{Tr}\left[\left(I - \Lambda^{B_1^n}_{X^n(1,1)}\right) \rho^{B^n}_{X^n(1,1)}\right]$$

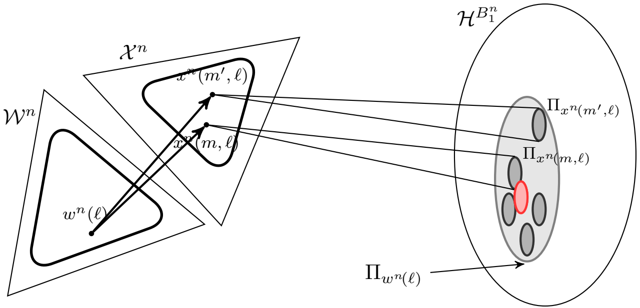
$$\leq 2 \underbrace{\text{Tr}\left[(I - P_{1,1})\ \rho^{B^n}_{X^n(1,1)}\right]}_{\text{Type I error}} + \underbrace{4 \sum_{(m,\ell)\neq(1,1)} \text{Tr}\left[P_{m,\ell}\ \rho^{B^n}_{X^n(1,1)}\right]}_{\text{Type II error}}.$$

Furthermore, the Type II error can be split into:

$$\sum_{(m,\ell)\neq(1,1)} \text{Tr}\left[P_{m,\ell}\ \rho^{B^n}_{X^n(1,1)}\right] = \underbrace{\sum_{m\neq 1} \text{Tr}\left[P_{m,1}\ \rho^{B_1}_{X^n(1,1)}\right]}_{\text{Error Type II (a)}} + \underbrace{\sum_{\substack{m,\\ \ell\neq 1}} \text{Tr}\left[P_{m,\ell}\ \rho^{B_1}_{X^n(1,1)}\right]}_{\text{Error Type II (b)}}.$$

Packing argument: Type II(a) error

Assume ℓ (the common message) was decoded correctly.

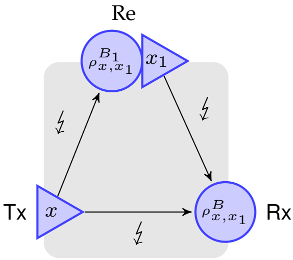


- The size of the $w^n(\ell)$-conditionally typical space is $\approx 2^{nH(B_1|W)}$.
- The size of each $x^n(m,\ell)$–conditionally typical "cloud" is $\approx 2^{nH(B_1|X,W)}$.
- Therefore, we can pack:

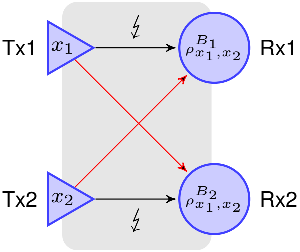
$$\frac{2^{nH(B_1|W)}}{2^{nH(B_1|X,W)}} = 2^{n[H(B_1|W)-H(B_1|X,W)]} = 2^{nI(X;B_1|W)},$$

codewords in the available space.

Quantum network-information theory



(a) QRC $\equiv \left\{ \rho^{B_1 B}_{x,x_1} \right\}$



(b) QIC $\equiv \left\{ \rho^{B_1 B_2}_{x_1,x_2} \right\}$

Figure: Other classical-quantum channels for which will be discussed in this session: (a) quantum relay channels (QRCs) and (b) quantum interference channels (QICs).

Each of these problems requires different error analysis techniques.

The end

Thank you for your attention!

See arXiv:1111.3645 for further details.