Attempting to reverse the irreversible in quantum physics

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Main message

- Entropy inequalities established in the 1970s are a mathematical consequence of the postulates of quantum physics
- They are helpful in determining the ultimate limits on many physical processes
- Many of these entropy inequalities are equivalent to each other, so we can say that together they constitute a fundamental law of quantum information theory
- There has been recent interest in refining these inequalities, trying to understand how well one can attempt to reverse an irreversible physical process
- This talk will discuss progress in this direction

Background — quantum mechanics

Quantum states

The state of a quantum system is specified by a positive semidefinite operator with trace equal to one, usually denoted by ρ, σ, τ, etc.

Quantum channels

Any physical process can be written as a quantum channel. Mathematically, a quantum channel is specified by a linear, completely positive, trace preserving map, so that it takes an input quantum state to an output quantum state. Quantum channels are usually denoted by $\mathcal{N}, \mathcal{M}, \mathcal{P}$, etc.

Quantum measurements

A quantum measurement is a special type of quantum channel with quantum input and classical output

Background — entropies

Umegaki relative entropy [Ume62]

The quantum relative entropy is a measure of dissimilarity between two quantum states. Defined for states ρ and σ as

$$D(\rho\|\sigma) \equiv \mathrm{Tr}\{\rho[\log \rho - \log \sigma]\}$$

whenever $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and $+\infty$ otherwise

Physical interpretation with quantum Stein's lemma [HP91, NO00]

Given are n quantum systems, all of which are prepared in either the state ρ or σ. With a constraint of $\varepsilon \in (0,1)$ on the Type I error of misidentifying ρ, then the optimal error exponent for the Type II error of misidentifying σ is $D(\rho\|\sigma)$.

Background — entropies

Relative entropy as "mother" entropy

Many important entropies can be written in terms of relative entropy:

- $H(A)_\rho \equiv -D(\rho_A \| I_A)$ (entropy)
- $H(A|B)_\rho \equiv -D(\rho_{AB} \| I_A \otimes \rho_B)$ (conditional entropy)
- $I(A;B)_\rho \equiv D(\rho_{AB} \| \rho_A \otimes \rho_B)$ (mutual information)
- $I(A;B|C)_\rho \equiv D(\rho_{ABC} \| \exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_C\})$ (cond. MI)

Equivalences

- $H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$
- $I(A;B)_\rho = H(A)_\rho + H(B)_\rho - H(AB)_\rho$
- $I(A;B)_\rho = H(B)_\rho - H(B|A)_\rho$
- $I(A;B|C)_\rho = H(AC)_\rho + H(BC)_\rho - H(ABC)_\rho - H(C)_\rho$
- $I(A;B|C)_\rho = H(B|C)_\rho - H(B|AC)_\rho$

Fundamental law of quantum information theory

Monotonicity of quantum relative entropy [Lin75, Uhl77]

Let ρ and σ be quantum states and let $\mathcal{N}$ be a quantum channel. Then

$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

"Distinguishability does not increase under a physical process"
Characterizes a fundamental irreversibility in any physical process

Proof approaches

- Lieb concavity theorem [L73]
- relative modular operator method (see, e.g., [NP04])
- quantum Stein's lemma [BS03]

Strong subadditivity [LR73]

Let ρ_{ABC} be a tripartite quantum state. Then

$$I(A;B|C)_\rho \geq 0$$

Equivalent statements (by definition)

- Entropy sum of two individual systems is larger than entropy sum of their union and intersection:

$$H(AC)_\rho + H(BC)_\rho \geq H(ABC)_\rho + H(C)_\rho$$

- Conditional entropy does not decrease under the loss of system A:

$$H(B|C)_\rho \geq H(B|AC)_\rho$$

Other equivalent entropy inequalities

Monotonicity of relative entropy under partial trace

Let ρ_{AB} and σ_{AB} be quantum states. Then $D(\rho_{AB}\|\sigma_{AB}) \geq D(\rho_B\|\sigma_B)$

Joint convexity of relative entropy

Let p_X be a probability distribution and let $\{\rho^x\}$ and $\{\sigma^x\}$ be sets of quantum states. Let $\overline{\rho} \equiv \sum_x p_X(x)\rho^x$ and $\overline{\sigma} \equiv \sum_x p_X(x)\sigma^x$. Then

$$\sum_x p_X(x) D(\rho^x\|\sigma^x) \geq D(\overline{\rho}\|\overline{\sigma})$$

Concavity of conditional entropy

Let p_X be a probability distribution and let $\{\rho^x_{AB}\}$ be a set of quantum states. Let $\overline{\rho}_{AB} \equiv \sum_x p_X(x)\rho^x_{AB}$. Then

$$H(A|B)_{\overline{\rho}} \geq \sum_x p_X(x) H(A|B)_{\rho^x}$$

1. Strong subadditivity $\Longrightarrow$ 2. Concavity of conditional entropy

$\Uparrow$ $\Downarrow$

5. Monotonicity of relative entropy

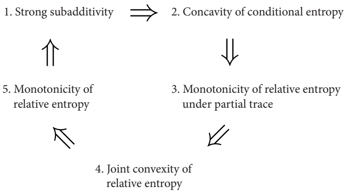
3. Monotonicity of relative entropy under partial trace

$\nwarrow$ $\swarrow$

4. Joint convexity of relative entropy

(discussed in [Rus02])

How to establish circle of equivalences?



- $(1 \Rightarrow 2)$ pick A to be a classical system in $I(A;B|C) \geq 0$.
- $(2 \Rightarrow 3)$ Apply 2 to $\{(\frac{1}{x+1}, \sigma_{AB}), (\frac{x}{x+1}, \rho_{AB})\}$, multiply by $\frac{x+1}{x}$, and take $\lim_{x \searrow 0}$
- $(3 \Rightarrow 4)$ Take A classical in $D(\rho_{AB} \| \sigma_{AB}) \geq D(\rho_B \| \sigma_B)$
- $(4 \Rightarrow 5)$ Stinespring, unitary averaging, and properties of $D(\rho \| \sigma)$
- $(5 \Rightarrow 1)$ Choose $\rho = \omega_{ABC}$, $\sigma = \omega_{AC} \otimes \omega_B$, and $\mathcal{N} = \mathrm{Tr}_A$ in $D(\rho \| \sigma) \geq D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma))$

Equality conditions

When does equality in monotonicity of relative entropy hold?

- $D(\rho\|\sigma) = D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$ iff $\exists$ a recovery map $\mathcal{R}^P_{\sigma,\mathcal{N}}$ such that

$$\rho = (\mathcal{R}^P_{\sigma,\mathcal{N}} \circ \mathcal{N})(\rho), \qquad \sigma = (\mathcal{R}^P_{\sigma,\mathcal{N}} \circ \mathcal{N})(\sigma)$$

- This “Petz” recovery map has the following explicit form [HJPW04]:

$$\mathcal{R}^P_{\sigma,\mathcal{N}}(\omega) \equiv \sigma^{1/2}\mathcal{N}^\dagger\left((\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right)\sigma^{1/2}$$

- Classical case: Distributions p_X and q_X and a channel $\mathcal{N}(y|x)$. Then the Petz recovery map $\mathcal{R}^P(x|y)$ is given by the Bayes theorem:

$$\mathcal{R}^P(x|y)q_Y(y) = \mathcal{N}(y|x)q_X(x)$$

where $q_Y(y) \equiv \sum_x \mathcal{N}(y|x)q_X(x)$

Pretty good measurement is special case of Petz recovery

- Take $\sigma = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \sigma^x$ and $\mathcal{N} = \mathrm{Tr}_X$. Then Petz recovery map is the "pretty good instrument" for recovering system X back:

$$\mathcal{R}^P_{\sigma,\mathcal{N}}(\cdot) = \sum_x |x\rangle\langle x|_X \otimes p_X(x)(\sigma^x)^{1/2}\overline{\sigma}^{-1/2}(\cdot)\overline{\sigma}^{-1/2}(\sigma^x)^{1/2}$$

  where $\overline{\sigma} = \sum_x p_X(x)\sigma^x$.

- Pretty good measurement map results from tracing over quantum system:

$$(\cdot) \to \sum_x \mathrm{Tr}\left\{\overline{\sigma}^{-1/2} p_X(x)\sigma^x \overline{\sigma}^{-1/2}(\cdot)\right\} |x\rangle\langle x|_X$$

More on Petz recovery map

- Linear, completely positive by inspection and trace preserving because

$$\begin{aligned}
\mathrm{Tr}\{\mathcal{R}^{P}_{\sigma,\mathcal{N}}(\omega)\} &= \mathrm{Tr}\{\sigma^{1/2}\mathcal{N}^{\dagger}\left((\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right)\sigma^{1/2}\} \\
&= \mathrm{Tr}\{\sigma\mathcal{N}^{\dagger}\left((\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right)\} \\
&= \mathrm{Tr}\{\mathcal{N}(\sigma)(\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\} \\
&= \mathrm{Tr}\{\omega\}
\end{aligned}$$

- Perfectly recovers σ from $\mathcal{N}(\sigma)$ because

$$\begin{aligned}
\mathcal{R}^{P}_{\sigma,\mathcal{N}}(\mathcal{N}(\sigma)) &= \sigma^{1/2}\mathcal{N}^{\dagger}\left((\mathcal{N}(\sigma))^{-1/2}\mathcal{N}(\sigma)(\mathcal{N}(\sigma))^{-1/2}\right)\sigma^{1/2} \\
&= \sigma^{1/2}\mathcal{N}^{\dagger}(I)\,\sigma^{1/2} \\
&= \sigma
\end{aligned}$$

Even more on Petz recovery map

Normalization [LW14]

For identity channel, the Petz recovery map is the identity map:

$$\mathcal{R}^{P}_{\sigma,\text{id}} = \text{id}$$

"If there's no noise, then no need to recover"

Tensorial [LW14]

Given a tensor-product state and channel, then the Petz recovery map is a tensor product of individual Petz recovery maps:

$$\mathcal{R}^{P}_{\sigma_1\otimes\sigma_2,\mathcal{N}_1\otimes\mathcal{N}_2} = \mathcal{R}^{P}_{\sigma_1,\mathcal{N}_1} \otimes \mathcal{R}^{P}_{\sigma_2,\mathcal{N}_2}$$

"Individual action suffices for 'pretty good' recovery of individual states"

And even more on Petz recovery map

Composition [LW14]

Given a composition of channels $\mathcal{C} \equiv \mathcal{N}_2 \circ \mathcal{N}_1$, then

$$\mathcal{R}^P_{\sigma,\mathcal{N}_2 \circ \mathcal{N}_1} = \mathcal{R}^P_{\sigma,\mathcal{N}_1} \circ \mathcal{R}^P_{\mathcal{N}_1(\sigma),\mathcal{N}_2}$$

"To recover 'pretty well' overall, recover 'pretty well' from the last noise first and the first noise last"

Short proof

Follows from inspection of definitions:

$$\mathcal{R}^P_{\sigma,\mathcal{N}_2 \circ \mathcal{N}_1}(\cdot) = \sigma^{1/2}\mathcal{C}^\dagger \left((\mathcal{C}(\sigma))^{-1/2}(\cdot)(\mathcal{C}(\sigma))^{-1/2} \right) \sigma^{1/2}$$

$$\mathcal{R}^P_{\sigma,\mathcal{N}_1} = \sigma^{1/2}\mathcal{N}_1^\dagger \left((\mathcal{N}_1(\sigma))^{-1/2}(\cdot)(\mathcal{N}_1(\sigma))^{-1/2} \right) \sigma^{1/2}$$

$$\mathcal{R}^P_{\mathcal{N}_1(\sigma),\mathcal{N}_2} = (\mathcal{N}_1(\sigma))^{1/2}\mathcal{N}_2^\dagger \left((\mathcal{C}(\sigma))^{-1/2}(\cdot)(\mathcal{C}(\sigma))^{-1/2} \right) (\mathcal{N}_1(\sigma))^{1/2}$$

Petz recovery map for strong subadditivity

- Recall that strong subadditivity is a special case of monotonicity of relative entropy with $\rho = \omega_{ABC}$, $\sigma = \omega_{AC} \otimes \omega_B$, and $\mathcal{N} = \mathrm{Tr}_A$
- Then $\mathcal{N}^\dagger(\cdot) = (\cdot) \otimes I_A$ and Petz recovery map is

$$\mathcal{R}^P_{C \to AC}(\tau_C) = \omega_{AC}^{1/2} \left(\omega_C^{-1/2} \tau_C \omega_C^{-1/2} \otimes I_A \right) \omega_{AC}^{1/2}$$

- Interpretation: If system A is lost but $H(B|C)_\omega = H(B|AC)_\omega$, then one can recover the full state on ABC by performing the Petz recovery map on system C of ω_{BC}, i.e.,

$$\omega_{ABC} = \mathcal{R}^P_{C \to AC}(\omega_{BC})$$

- Exact result [HJPW04]: $H(B|C)_\omega = H(B|AC)_\omega$ iff ω_{ABC} is a quantum Markov state, i.e., $\exists$ a decomposition of C, a distribution p_Z and sets $\{\tau^z_{AC^{L_z}}\}$, $\{\tau^z_{C^{R_z}B}\}$ of states such that

$$\omega_{ABC} = \bigoplus_z p_Z(z) \tau^z_{AC^{L_z}} \otimes \tau^z_{C^{R_z}B}$$

Approximate case

Approximate case would be useful for applications

Approximate case for monotonicity of relative entropy

- What can we say when $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = \varepsilon$?
- Does there exist a CPTP map $\mathcal{R}$ that recovers σ perfectly from $\mathcal{N}(\sigma)$ while recovering ρ from $\mathcal{N}(\rho)$ approximately? [WL12]

Approximate case for strong subadditivity

- What can we say when $H(B|C)_\omega - H(B|AC)_\omega = \varepsilon$?
- Is ω_{ABC} close to a quantum Markov state? [ILW08]
- Is ω_{ABC} approximately recoverable from ω_{BC} by performing a recovery map on system C alone? [WL12]

No-go result for closeness to q. Markov states [ILW08]

- Define relative entropy to quantum Markov states as

$$\Delta(\omega_{ABC}) \equiv \min_{\sigma_{ABC} \in \mathcal{M}_{A-C-B}} D(\omega_{ABC} \| \sigma_{ABC})$$

- It is known that there exist states ω_{ABC} for which

$$\Delta(\omega_{ABC}) \gg I(A;B|C)_{\omega}$$

- Very different from classical case. Conclusion is that closeness to quantum Markov states does not characterize states with small conditional mutual information
- Other possibility is in terms of recoverability...

Other measures of similarity for quantum states

Trace distance

Trace distance between ρ and σ is $\|\rho - \sigma\|_1$ where $\|A\|_1 = \text{Tr}\{\sqrt{A^\dagger A}\}$.
Has a one-shot operational interpretation as the bias in success probability when distinguishing ρ and σ with an optimal quantum measurement.

Fidelity [Uhl76]

Fidelity between ρ and σ is $F(\rho, \sigma) \equiv \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$. Has a one-shot operational interpretation as the probability with which a purification of ρ could pass a test for being a purification of σ.

Bures distance [Bur69]

Bures distance between ρ and σ is $D_B(\rho, \sigma) = \sqrt{2(1 - \sqrt{F(\rho, \sigma)}}$.

Conjectures for the approximate case

Conjecture for monotonicity of relative entropy [SBW14]

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\log F\left(\rho, \mathcal{R}^P_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))\right)$$
$$\geq D_B^2\left(\rho, \mathcal{R}^P_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))\right)$$

Conjecture for strong subadditivity [BSW14]

$$H(B|C)_\omega - H(B|AC)_\omega \geq -\log F\left(\omega_{ABC}, \mathcal{R}^P_{C\to AC}(\omega_{BC})\right)$$
$$\geq D_B^2\left(\omega_{ABC}, \mathcal{R}^P_{C\to AC}(\omega_{BC})\right)$$

- Would follow from conjectures regarding Rényi relative entropy differences and Rényi conditional mutual information (see related conjectures in [WL12, Kim13, Zha14])

Breakthrough result of [FR14]

Remainder term for strong subadditivity [FR14]

$\exists$ unitary channels $\mathcal{U}_C$ and $\mathcal{V}_{AC}$ such that

$$H(B|C)_\omega - H(B|AC)_\omega \geq -\log F\left(\omega_{ABC}, (\mathcal{V}_{AC} \circ \mathcal{R}^P_{C\to AC} \circ \mathcal{U}_C)(\omega_{BC})\right)$$

Nothing known about these unitaries! However, can conclude that $I(A;B|C)$ is small iff ω_{ABC} is approximately recoverable from system C alone after the loss of system A. Gives a good notion of approximate quantum Markov chain...

Remainder term for monotonicity of relative entropy [BLW14]

$\exists$ unitary channels $\mathcal{U}$ and $\mathcal{V}$ such that

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\log F\left(\rho, (\mathcal{V} \circ \mathcal{R}^P_{\sigma,\mathcal{N}} \circ \mathcal{U})(\mathcal{N}(\rho))\right)$$

Again, nothing known about $\mathcal{U}$ and $\mathcal{V}$. Furthermore, unclear how this "rotated Petz map" performs when recovering σ from $\mathcal{N}(\sigma)$

Implications of [FR14]

When quantum discord is nearly equal to zero [SW14]

(Unoptimized) quantum discord of ρ_{AB} defined as

$$I(A;B)_\rho - I(X;B)_\tau$$

where

$$\tau_{XB} \equiv \sum_x |x\rangle\langle x| \otimes \text{Tr}_A\{\Lambda_A^x \rho_{AB}\}$$

Then

$$I(A;B)_\rho - I(X;B)_\tau \geq -\log F(\rho_{AB}, \mathcal{E}_A(\rho_{AB}))$$

where $\mathcal{E}_A$ is an entanglement breaking channel. Conclusion: discord is nearly equal to zero iff ρ_{AB} is approximately recoverable after performing a measurement on system A (equivalently, iff ρ_{AB} is an approximate fixed point of an entanglement breaking channel)

More implications of [FR14]

Approximate faithfulness of squashed entanglement [WL12, LW14]

Squashed entanglement of ρ_{AB} defined as

$$E^{\text{sq}}(A;B)_\rho \equiv \frac{1}{2} \inf_{\omega_{ABE}} \{ I(A;B|E)_\omega \,|\, \rho_{AB} = \text{Tr}_E\{\omega_{ABE}\} \}$$

Then

$$E^{\text{sq}}(A;B)_\rho \geq \frac{C}{\dim |B|^4} \left\| \rho_{AB} - \text{SEP}(A:B) \right\|_1^4$$

where C is a constant. Proof idea is to use the rotated Petz recovery map to extract several approximate copies of system A from E alone. Randomly permuting these gives a k-extension of the original state and one can approximate SEP with the set of k-extendible states.

Even more implications of [FR14]

Approximate faithfulness of multipartite squashed-like entanglement

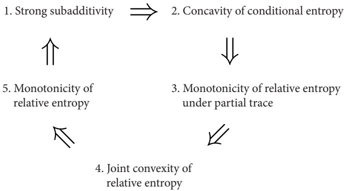
Can show a similar bound for the conditional entanglement of multipartite information (squashed-like measure from [YHW08]). Conclusion: This measure is faithful.

Multipartite discord

Multipartite discord of a multipartite state [PHH08] is nearly equal to zero if and only if each system is approximately locally recoverable after performing a measurement on each system.

- (see [Wil14] for details)

Refinement of the circle of equivalences [BLW14]



Can show that the circle of equivalences still holds with remainder terms given by Bures distance, e.g., the following implication is true:

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq D_B^2\left(\rho, \mathcal{R}^P_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))\right)$$

$$\Rightarrow I(A;B|C)_\omega \geq D_B^2\left(\omega_{ABC}, \mathcal{R}^P_{C\to AC}(\omega_{BC})\right)$$

and etc. Unknown if any single ineq. is true, so either all true or all false!

Conclusions

- The result of [FR14] already has a number of important implications in quantum information theory. However, it would be ideal to have the recovery map be the Petz recovery map alone (not a rotated Petz map).

- It seems that these refinements with Petz recovery should be true (at least numerical evidence and intuition supports). If so, we would have a strong refinement of the fundamental law of quantum information theory, characterizing how well one could attempt to reverse an irreversible process. We could also then say "the circle is now complete."

- Furthermore, there will be important implications in a number of fields, including quantum communication complexity, quantum information theory, thermodynamics, condensed matter physics

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