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Renyi Generalizations of Conditional Mutual Information

11 APR 2014

arXiv:1403.6102

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Given is a tripartite quantum state ρ_{ABC}, w/ marginals denoted by ρ_A, ρ_{AB}, etc. The conditional mutual information of A & B from the perspective of C is

$$I(A;B|C)_\rho = H(AC)_\rho + H(BC)_\rho - H(C)_\rho - H(ABC)_\rho$$

where $H(F)_\sigma = -\mathrm{Tr}\{\sigma_F \log \sigma_F\}$ for a state σ on system F.

This quantity shows up everywhere in quantum information theory: strong subadditivity, squashed entanglement, discord, etc.

Goal of this talk: Find a compelling Renyi generalization of this quantity.

Why? 1) Renyi entropies generalize von Neumann ones and give us a refined understanding of quantum information theory (examples: min- & max-entropy, smoothed versions, strong converses, error exponents, etc.)

2) A Renyi generalization might give us a way of attacking important open questions concerning $I(A;B|C)$ (they are fundamental questions but also have application in condensed matter phys. + QIT)

Before finding a Renyi generalization, we should determine some relevant properties that it should obey. So we turn to $I(A;B|C)$ for this.

1) <u>Non-negativity</u>

$$I(A;B|C)_\rho \geq 0 \qquad \forall \rho_{ABC}$$

This is the statement of strong subadditivity (Lieb & Ruskai 1973) used in nearly every coding theorem in QIT & is one reason why squashed entanglement is a sensible correlation measure.

2) <u>Data processing under local ops on A & B</u>

$$I(A_1A_2;B_1|C) \geq I(A_1;B_1|C)$$

Why is this true? just write it out

$$\Leftrightarrow H(A_1A_2C) + H(B_1C) - H(C) - H(A_1A_2B_1C) \geq H(A_1C) + H(B_1C) - H(C) - H(A_1B_1C)$$

$$\Leftrightarrow H(A_1A_2C) - H(A_1A_2B_1C) \geq H(A_1C) - H(A_1B_1C)$$

$$\Leftrightarrow H(A_1A_2C) + H(A_1B_1C) - H(A_1C) - H(A_1A_2B_1C) \geq 0$$

$$\Leftrightarrow I(A_2;B_1|A_1C) \geq 0 \quad \text{by SSA}$$

symmetric argument implies

$$I(A_1;B_1B_2|C) \geq I(A_1;B_1|C)$$

so we get

$$I(A_1A_2;B_1B_2|C) \geq I(A_1;B_1|C)$$

3) <u>Duality for 4-party pure states</u>

Given pure ψ_{ABCD}, we have that $I(A;B|C)_\psi = I(A;B|D)_\psi$

So if we make a Renyi generalization of CMI, we would like for it to obey properties like the above ones.

<u>1st bad attempt:</u>

Let $I'_\alpha(A;B|C)_\rho =$

$$H_\alpha(AC)_\rho + H_\alpha(BC)_\rho - H_\alpha(C)_\rho - H_\alpha(ABC)_\rho$$

where $H_\alpha(F)_\sigma = \frac{1}{1-\alpha}\log \mathrm{Tr}\{\sigma_F^\alpha\}$

& $\alpha \in [0,\infty]$

Observe that $\alpha \to 1$ gives $H(F)_\sigma$

(need L'Hôpital's rule for this)
& Taylor expand about $\alpha=1$

It is true that $I'_\alpha(A;B|C)_\rho = I'_\alpha(A;B|D)_\rho$

but non-negativity & data processing

do not hold in general!

(see arXiv:1212.0248 Linden, Mosonyi, Winter.

This paper shows that there are generally no constraints placed on linear combinations of Renyi entropies)

Since these properties are used in applications all the time, we should essentially discard I'_α as a useful generalization.

So what should we do instead?

Consider that all known Renyi generalizations of information quantities make use of the relative entropy $D(\rho\|\sigma) = \mathrm{Tr}\{\rho \log \rho\} - \mathrm{Tr}\{\rho \log \sigma\}$

That is
$$H(A)_\rho = -D(\rho_A \| I_A)$$
$$H(A|B)_\rho = -D(\rho_{AB} \| I_A \otimes \rho_B)$$
$$I(A;B)_\rho = D(\rho_{AB} \| \rho_A \otimes \rho_B)$$

So we should figure out how to write $I(A;B|C)_\rho$ as a relative entropy.

Using the definition, we find

$$\begin{aligned} I(A;B|C) = &-\mathrm{Tr}\{\rho_{AC} \log \rho_{AC}\} \\ &-\mathrm{Tr}\{\rho_{BC} \log \rho_{BC}\} \\ &+\mathrm{Tr}\{\rho_C \log \rho_C\} \\ &+\mathrm{Tr}\{\rho_{ABC} \log \rho_{ABC}\} \end{aligned}$$

$$= \mathrm{Tr}\{\rho_{ABC} \log \rho_{ABC}\} - \mathrm{Tr}\{\rho_{ABC} [\log \rho_{AC} + \log \rho_{BC} - \log \rho_C]\}$$

$$= \mathrm{Tr}\{\rho_{ABC} \log \rho_{ABC}\}$$
$$- \mathrm{Tr}\{\rho_{ABC} \log [\exp(\log \rho_{AC} + \log \rho_{BC} - \log \rho_C)]\}$$
$$= D(\rho_{ABC} \| \underbrace{\exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_C\}}_{})$$

this operator is interesting
it is non-negative &
in fact subnormalized
w/ $\mathrm{Tr}\{\cdot\} \leq 1$

(from triple matrix generalization
of Golden-Thompson
inequality from
stat. mech.)

We know that the Renyi relative entropy is

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1} \log \mathrm{Tr}\{\rho^\alpha \sigma^{1-\alpha}\}$$

there is also the "sandwiched" version

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1} \log \mathrm{Tr}\{(\sigma^{\frac{1-\alpha}{2\alpha}} \rho\, \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha\}$$

For these, we have $\lim_{\alpha \to 1} D_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$

So we might guess ~~that~~ to take α-CMI as

$$D_\alpha(\rho_{ABC} \| \exp\{\log \cdots\}),$$

but there is a better choice.

The Lie-Trotter product formula gives that

$$\exp\{\log\rho_{AC} + \log\rho_{BC} - \log\rho_C\}$$

$$= \lim_{\alpha\to 1} \underbrace{\left(\rho_{AC}^{\frac{1-\alpha}{2}}\rho_C^{\frac{\alpha-1}{2}}\rho_{BC}^{1-\alpha}\rho_C^{\frac{\alpha-1}{2}}\rho_{AC}^{\frac{1-\alpha}{2}}\right)}_{}^{\frac{1}{1-\alpha}}$$

this could be any operator ordering, but we take them such that resulting operator is positive.

So we define

$$I_\alpha(A;B|C)_{\rho|\rho} = \frac{1}{\alpha-1}\log\mathrm{Tr}\{\rho_{ABC}^{\alpha}\,\rho_{AC}^{\frac{1-\alpha}{2}}\rho_C^{\frac{\alpha-1}{2}}\rho_{BC}^{1-\alpha} \times \rho_C^{\frac{\alpha-1}{2}}\rho_{AC}^{\frac{1-\alpha}{2}}\}$$

We can prove that

$$\lim_{\alpha\to 1} I_\alpha(A;B|C)_{\rho|\rho} = I(A;B|C)_\rho$$ ✓ by L'Hôpital's & Taylor expanding

Data processing under local operations on B for $\alpha \in [0,2]$

For this, it suffices to prove that

$$\mathrm{Tr}\left\{ \rho_{ABC}^{\alpha}\, \rho_{AC}^{\frac{1-\alpha}{2}}\, \rho_{C}^{\frac{\alpha-1}{2}}\, \rho_{BC}^{1-\alpha}\, \rho_{C}^{\frac{\alpha-1}{2}}\, \rho_{AC}^{\frac{1-\alpha}{2}} \right\} \quad (*)$$

is

1) jointly concave in ρ_{ABC} + ρ_{BC} for $\alpha \in [0,1]$

open question → 2) " " in ρ_{ABC} + ρ_{AC} "

3) jointly convex in ρ_{ABC} + ρ_{BC} for $\alpha \in [1,2]$

open question → 4) " " in ρ_{ABC} + ρ_{AC} for $\alpha \in [1,2]$

To get 1), We know the Lieb concavity theorem, that

$$(S,R) \rightarrow \mathrm{Tr}\{ S^{\gamma} X R^{1-\gamma} X^{\dagger} \}$$

is jointly concave in $S + R$

for $\gamma \in [0,1]$

for $S, R \geq 0$ + X arbitrary

So set $S = \rho_{ABC}$, $R = \rho_{BC}$

+ $X = \rho_{AC}^{\frac{1-\alpha}{2}}\, \rho_{C}^{\frac{\alpha-1}{2}}$

Can define sandwiched Renyi CMI as

$$\tilde{I}_\alpha(A;B|C)_{\rho|\rho} = \tilde{D}_\alpha\left(\rho_{ABC} \,\Big\|\, \left(\rho_{AC}^{\frac{1-\alpha}{2\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{BC}^{\frac{1-\alpha}{\alpha}} \rho_C^{\frac{\alpha-1}{2\alpha}} \rho_{AC}^{\frac{1-\alpha}{2\alpha}}\right)^{\frac{\alpha}{1-\alpha}}\right)$$

can show that this one obeys data processing under local ops on B for $\alpha \in [1/2, \infty]$

data processing under local ops gives non-negativity (for α for which DP holds)

Why? Take local ops to be trace out maps

Then

$$I_\alpha(A;B|C)_{\rho|\rho} = D_\alpha\left(\rho_{ABC} \,\Big\|\, \left\{\rho_{AC}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_{BC}^{1-\alpha} \rho_C^{\frac{\alpha-1}{2}} \rho_{AC}^{\frac{1-\alpha}{2}}\right\}^{\frac{1}{1-\alpha}}\right)$$

$$\geq D_\alpha\left(\rho_{AC} \,\Big\|\, \left\{\rho_{AC}^{\frac{1-\alpha}{2}} \rho_C^{\frac{\alpha-1}{2}} \rho_C^{1-\alpha} \rho_C^{\frac{\alpha-1}{2}} \rho_{AC}^{\frac{1-\alpha}{2}}\right\}^{\frac{1}{1-\alpha}}\right)$$

$$= D_\alpha(\rho_{AC} \| \rho_{AC}) = 0$$

<u>Big conjecture:</u> Monotonicity in α

I.e., $\forall\ 0 \le \alpha \le \beta$, we have

$$I_\alpha(A;B|C)_{\rho|\rho} \le I_\beta(A;B|C)_{\rho|\rho}$$

$$+ \quad \widetilde{I}_\alpha(\ldots) \le \widetilde{I}_\beta(\ldots)$$

lots of numerical evidence +
a proof for α in a neighborhood of 1.

<u>Idea for proof:</u> if derivative of I_α wrt α is non-negative, then the function is monotone increasing.

Can show that

$$\frac{d}{d\alpha} I_\alpha = \underbrace{\frac{(\alpha-1)^2}{2} V(A;B|C)}_{\text{something positive}} + O((\alpha-1)^3)$$

where $V(A;B|C) =$

$$\mathrm{Tr}\{\rho_{ABC}[\log \rho_{ABC} + \log \rho_C - \log \rho_{AC} - \log \rho_{BC} - I(A;B|C)]^2\}$$

$$\ge 0$$

Conjecture implies that

$$I(A;B|C)_\rho \geq \frac{1}{4} \| \rho_{ABC} - \rho_{AC}^{1/2} \rho_C^{-1/2} \rho_{BC} \rho_C^{-1/2} \rho_{AC}^{1/2} \|$$

So it would mean that if

$I(A;B|C)_\rho \leq \varepsilon$ then

if you have ρ_{BC} the

"pretty good" recovery map

$$R_{BC \to ABC}(\cdot) = \rho_{AC}^{1/2} \rho_C^{-1/2} (\cdot) \rho_C^{-1/2} \rho_{AC}^{1/2}$$

is good for recovering ρ_{ABC}

from ρ_{BC} alone.

Final point: the method given here extends beyond just CMI. It gives a Renyi generalization for any information quantity that can be realized as a linear combination of vN entropies w/ coefficients taken from $\{-1, 0, +1\}$

Open questions: Solve conjecture & make use of this in applications.