Recoverability in quantum information theory

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Entropy inequalities established in the 1970s are a mathematical consequence of the postulates of quantum physics. They are helpful in determining the ultimate limits on many physical processes (communication, thermodynamics, uncertainty relations). Many of these entropy inequalities are equivalent to each other, so we can say that together they constitute a fundamental law of quantum information theory. There has been recent interest in refining these inequalities, trying to understand how well one can attempt to reverse an irreversible physical process. This talk discusses progress in this direction.
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Umegaki relative entropy \[\text{[Ume62]}\]

The quantum relative entropy is a measure of dissimilarity between two quantum states. Defined for state \(\rho\) and positive semi-definite \(\sigma\) as

\[
D(\rho \parallel \sigma) \equiv \text{Tr}\left\{ \rho \left[ \log \rho - \log \sigma \right] \right\}
\]

whenever \(\text{supp}(\rho) \subseteq \text{supp}(\sigma)\) and \(+\infty\) otherwise

Physical interpretation with quantum Stein's lemma \[\text{[HP91, NO00]}\]

Given are \(n\) quantum systems, all of which are prepared in either the state \(\rho\) or \(\sigma\). With a constraint of \(\varepsilon \in (0, 1)\) on the Type I error of misidentifying \(\rho\), then the optimal error exponent for the Type II error of misidentifying \(\sigma\) is \(D(\rho \parallel \sigma)\).
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Background — entropies

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Background — entropies

Relative entropy as “mother” entropy

Many important entropies can be written in terms of relative entropy:

\[ H(A)_{\rho} \equiv -D(\rho A || I_A) \] (entropy)

\[ H(A|B)_{\rho} \equiv -D(\rho_{AB} || I_A \otimes \rho_B) \] (conditional entropy)

\[ I(A;B)_{\rho} \equiv D(\rho_{AB} || \rho_A \otimes \rho_B) \] (mutual information)

\[ I(A;B|C)_{\rho} \equiv D(\rho_{ABC} || \exp\{\log \rho_{AC} + \log \rho_{BC} - \log \rho_C\}) \] (cond. MI)

Equivalences

\[ H(A|B)_{\rho} = H(AB)_{\rho} - H(B)_{\rho} \]

\[ I(A;B)_{\rho} = H(A)_{\rho} + H(B)_{\rho} - H(AB)_{\rho} \]

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# Relative entropy as “mother” entropy

Many important entropies can be written in terms of relative entropy:

- **Entropy of a system**:
  \[ H(A)_{\rho} \equiv -D(\rho_A \parallel I_A) \]

- **Conditional entropy**:
  \[ H(A|B)_{\rho} \equiv -D(\rho_{AB} \parallel I_A \otimes \rho_B) \]

- **Mutual information**:
  \[ I(A;B)_{\rho} \equiv D(\rho_{AB} \parallel \rho_A \otimes \rho_B) \]

- **Conditional mutual information**:
  \[ I(A;B|C)_{\rho} \equiv D(\rho_{ABC} \parallel \rho_{AC} \otimes \rho_{BC} - \log \rho_C) \]

Equivalences:

- **Conditional entropy to mutual information**:
  \[ H(A|B)_{\rho} = H(AB)_{\rho} - H(B)_{\rho} \]

- **Mutual information to entropies**:
  \[ I(A;B)_{\rho} = H(A)_{\rho} + H(B)_{\rho} - H(AB)_{\rho} \]

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Monotonicity of quantum relative entropy \[\text{Lin75, Uhl77}\]

Let \(\rho\) be a state, let \(\sigma\) be positive semi-definite, and let \(\mathcal{N}\) be a quantum channel. Then

\[
D(\rho \parallel \sigma) \geq D(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma))
\]

"Distinguishability does not increase under a physical process"

Characterizes a fundamental irreversibility in any physical process

Proof approaches

- Lieb concavity theorem [L73]
- Relative modular operator method (see, e.g., [NP04])
- Quantum Stein's lemma [BS03]
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Fundamental law of quantum information theory

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Strong subadditivity

Let $\rho_{ABC}$ be a tripartite quantum state. Then

$$I(A;B|C)_{\rho} \geq 0$$

Equivalent statements (by definition)

Entropy sum of two individual systems is larger than entropy sum of their union and intersection:

$$H(AC)_{\rho} + H(BC)_{\rho} \geq H(ABC)_{\rho} + H(C)_{\rho}$$

Conditional entropy does not decrease under the loss of system $A$:

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Equality conditions [Pet86, Pet88]

When does equality in monotonicity of relative entropy hold? 

\[ D(\rho \| \sigma) = D(N(\rho) \| N(\sigma)) \text{ iff } \exists \text{ a recovery map } R_{\mathcal{P}\sigma, N} \]

This "Petz" recovery map has the following explicit form [HJPW04]:

\[ R_{\mathcal{P}\sigma, N}(\omega) \equiv \frac{1}{2} N^\dagger (N(\sigma) - \frac{1}{2} \omega (N(\sigma))^{-1/2}) \frac{1}{2} \]

Classical case: Distributions \( p_X \) and \( q_X \) and a channel \( N(y|x) \). Then the Petz recovery map \( R_{\mathcal{P}}(x|y) \) is given by the Bayes theorem:

\[ R_{\mathcal{P}}(x|y) = N(y|x) q_X(x) \]

where \( q_Y(y) \equiv \sum_x N(y|x) q_X(x) \)
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When does equality in monotonicity of relative entropy hold?

\[ \text{D}(\rho \parallel \sigma) = \text{D}(N(\rho) \parallel N(\sigma)) \text{ iff } \exists \text{ a recovery map } R_{\sigma, N} \text{ such that } \rho = (R_{\sigma, N} \circ N)(\rho), \sigma = (R_{\sigma, N} \circ N)(\sigma) \]

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- \( D(\rho \| \sigma) = D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)) \) iff \( \exists \) a recovery map \( \mathcal{R}_{\sigma,\mathcal{N}}^p \) such that

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  where \(q_Y(y) \equiv \sum_x \mathcal{N}(y|x) q_X(x)\)
More on Petz recovery map

Linear, completely positive by inspection and trace non-increasing because

\[
\text{Tr} \{ R_{\sigma, N}(\omega) \} = \text{Tr} \{ \sigma \frac{1}{2} N^\dagger \left( (N(\sigma)) - \frac{1}{2} \omega (N(\sigma)) \right) \sigma \frac{1}{2} \} = \text{Tr} \{ N(\sigma) (N(\sigma)) - \frac{1}{2} \omega (N(\sigma)) \} \leq \text{Tr} \{ \omega \}
\]

For \( N(\sigma) \) positive definite, the map perfectly recovers \( \sigma \) from \( N(\sigma) \):

\[
R_{\sigma, N}(N(\sigma)) = \sigma \frac{1}{2} N^\dagger \left( (N(\sigma)) - \frac{1}{2} \omega (N(\sigma)) \right) \sigma \frac{1}{2} = \sigma \frac{1}{2} N^\dagger (I) \sigma \frac{1}{2} = \sigma
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Linear, completely positive by inspection and trace non-increasing because

\[ \text{Tr}\{R_{\sigma,N}^P(\omega)\} = \text{Tr}\{\sigma^{1/2}N^\dagger \left((N(\sigma))^{-1/2}\omega(N(\sigma))^{-1/2}\right)\sigma^{1/2}\} \]
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\[ = \text{Tr}\{N(\sigma)(N(\sigma))^{-1/2}\omega(N(\sigma))^{-1/2}\} \]
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More on Petz recovery map

- Linear, completely positive by inspection and trace non-increasing because

$$\text{Tr}\{\mathcal{R}_{\sigma,N}(\omega)\} = \text{Tr}\{\sigma^{1/2}N^\dagger\left[(\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right]\sigma^{1/2}\}$$

$$= \text{Tr}\{\sigma N^\dagger\left[(\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right]\}$$

$$= \text{Tr}\{\mathcal{N}(\sigma)(\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\}$$

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$$= \sigma^{1/2}N^\dagger(I)\sigma^{1/2}$$

$$= \sigma$$
Functoriality

For identity channel, the Petz recovery map is the identity map: $R_P\sigma, \text{id} = \text{id}$. "If there's no noise, then no need to recover"

Tensorial \[ \text{LW14} \]

Given a tensor-product state and channel, then the Petz recovery map is a tensor product: $R_P\sigma_1 \otimes \sigma_2, N_1 \otimes N_2 = R_P\sigma_1, N_1 \otimes R_P\sigma_2, N_2$. "Individual actions suffice for 'pretty good' recovery of individual states"

Composition \[ \text{LW14} \]

Given $N_2 \circ N_1$, then $R_P\sigma, N_2 \circ N_1 = R_P\sigma, N_1 \circ R_P N_1(\sigma), N_2$. "To recover 'pretty well' overall, recover 'pretty well' from the last noise first and the first noise last"
Functoriality

Normalization [LW14]

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Given $\mathcal{N}_2 \circ \mathcal{N}_1$, then $\mathcal{R}^P_{\sigma, \mathcal{N}_2 \circ \mathcal{N}_1} = \mathcal{R}^P_{\sigma, \mathcal{N}_1} \circ \mathcal{R}^P_{\mathcal{N}_1(\sigma), \mathcal{N}_2}$. “To recover ‘pretty well’ overall, recover ‘pretty well’ from the last noise first and the first noise last”
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Given a tensor-product state and channel, then the Petz recovery map is a tensor product: \( \mathcal{R}^P_{\sigma_1 \otimes \sigma_2, \mathcal{N}_1 \otimes \mathcal{N}_2} = \mathcal{R}^P_{\sigma_1, \mathcal{N}_1} \otimes \mathcal{R}^P_{\sigma_2, \mathcal{N}_2} \). “Individual action suffices for ‘pretty good’ recovery of individual states”

Composition [LW14]
Given \( \mathcal{N}_2 \circ \mathcal{N}_1 \), then \( \mathcal{R}^P_{\sigma, \mathcal{N}_2 \circ \mathcal{N}_1} = \mathcal{R}^P_{\sigma, \mathcal{N}_1} \circ \mathcal{R}^P_{\mathcal{N}_1(\sigma), \mathcal{N}_2} \). “To recover ‘pretty well’ overall, recover ‘pretty well’ from the last noise first and the first noise last”
Strong subadditivity is a special case of monotonicity of relative entropy with $\rho = \omega_{ABC}$, $\sigma = \omega_{AC} \otimes I_B$, and $N = \text{Tr}_A$. Then $N^*(\cdot) = (\cdot) \otimes I_A$ and Petz recovery map is $R_{PC} : AC(\tau_C) = \frac{1}{2} AC(\omega - \frac{1}{2} C \tau_C \omega - \frac{1}{2} C \otimes I_A)$.

Interpretation: If system $A$ is lost but $H(B|C) = H(B|AC)$, then one can recover the full state on $ABC$ by performing the Petz recovery map on system $C$ of $\omega_{BC}$, i.e., $\omega_{ABC} = R_{PC} : AC(\omega_{BC})$. 

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Strong subadditivity is a special case of monotonicity of relative entropy with \( \rho = \omega_{ABC} \), \( \sigma = \omega_{AC} \otimes I_B \), and \( \mathcal{N} = \text{Tr}_A \)

\[ N(\cdot) = (\cdot) \otimes I_A \]

Petz recovery map is \( R_{PC} \rightarrow AC(\tau_C) = \frac{1}{2} AC(\omega - \frac{1}{2} C \tau_C \omega - \frac{1}{2} C \otimes I_A) \frac{1}{2} AC \)

Interpretation: If system \( A \) is lost but \( H(B|C) = H(B|AC) \), then one can recover the full state on \( ABC \) by performing the Petz recovery map on system \( C \) of \( \omega_{BC} \), i.e.,

\[ \omega_{ABC} = R_{PC} \rightarrow AC(\omega_{BC}) \]
Strong subadditivity is a special case of monotonicity of relative entropy with \( \rho = \omega_{ABC} \), \( \sigma = \omega_{AC} \otimes I_B \), and \( \mathcal{N} = \text{Tr}_A \). Then \( \mathcal{N}^\dagger(\cdot) = (\cdot) \otimes I_A \) and Petz recovery map is

\[
R_{C \rightarrow AC}^P(\tau_C) = \omega_{AC}^{1/2} \left( \omega_C^{-1/2} \tau_C \omega_C^{-1/2} \otimes I_A \right) \omega_{AC}^{1/2}
\]

Interpretation: If system \( A \) is lost but \( H(B|C) = H(B|AC) \), then one can recover the full state on \( ABC \) by performing the Petz recovery map on system \( C \) of \( \omega_{BC} \), i.e., \( \omega_{ABC} = R_{C \rightarrow AC}^P(\omega_{BC}) \).
Petz recovery map for strong subadditivity

- Strong subadditivity is a special case of monotonicity of relative entropy with $\rho = \omega_{ABC}$, $\sigma = \omega_{AC} \otimes I_B$, and $N = \text{Tr}_A$

- Then $N^\dagger(\cdot) = (\cdot) \otimes I_A$ and Petz recovery map is

  $$\mathcal{R}_{C \rightarrow AC}^P(\tau_C) = \omega_{AC}^{1/2} \left( \omega_C^{-1/2} \tau_C \omega_C^{-1/2} \otimes I_A \right) \omega_{AC}^{1/2}$$

- Interpretation: If system $A$ is lost but $H(B|C)_{\omega} = H(B|AC)_{\omega}$, then one can recover the full state on $ABC$ by performing the Petz recovery map on system $C$ of $\omega_{BC}$, i.e.,

  $$\omega_{ABC} = \mathcal{R}_{C \rightarrow AC}^P(\omega_{BC})$$
Approximate case

Approximate case would be useful for applications
Approximate case

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Approximate case for monotonicity of relative entropy

What can we say when $D(\rho \parallel \sigma) - D(N(\rho) \parallel N(\sigma)) = \varepsilon$?

Does there exist a CPTP map $R$ that recovers $\sigma$ perfectly from $N(\sigma)$ while recovering $\rho$ from $N(\rho)$ approximately? [WL12]

Approximate case for strong subadditivity

What can we say when $H(B|C) - H(B|AC) = \varepsilon$?

Is $\omega_{ABC}$ approximately recoverable from $\omega_{BC}$ by performing a recovery map on system $C$ alone? [WL12]
Approximate case

Approximate case would be useful for applications

Approximate case for monotonicity of relative entropy

- What can we say when $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = \varepsilon$?
Approximate case would be useful for applications

**Approximate case for monotonicity of relative entropy**

- What can we say when \( D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = \varepsilon \)?
- Does there exist a CPTP map \( \mathcal{R} \) that recovers \( \sigma \) perfectly from \( \mathcal{N}(\sigma) \) while recovering \( \rho \) from \( \mathcal{N}(\rho) \) approximately? [WL12]
Approximate case

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Approximate case for monotonicity of relative entropy

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Approximate case for strong subadditivity

- What can we say when \( H(B|C)_\omega - H(B|AC)_\omega = \varepsilon \)?
Approximate case

Approximate case would be useful for applications

Approximate case for monotonicity of relative entropy

- What can we say when $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) = \varepsilon$?
- Does there exist a CPTP map $\mathcal{R}$ that recovers $\sigma$ perfectly from $\mathcal{N}(\sigma)$ while recovering $\rho$ from $\mathcal{N}(\rho)$ approximately? [WL12]

Approximate case for strong subadditivity

- What can we say when $H(B|C)_\omega - H(B|AC)_\omega = \varepsilon$?
- Is $\omega_{ABC}$ approximately recoverable from $\omega_{BC}$ by performing a recovery map on system $C$ alone? [WL12]
Other measures of similarity for quantum states

Trace distance

Trace distance between $\rho$ and $\sigma$ is $\|\rho - \sigma\|_1$ where $\|A\|_1 = \text{Tr}\{\sqrt{A^\dagger A}\}$.

Has a one-shot operational interpretation as the bias in success probability when distinguishing $\rho$ and $\sigma$ with an optimal quantum measurement.

Fidelity

Fidelity between $\rho$ and $\sigma$ is $F(\rho, \sigma) \equiv \|\sqrt{\rho} \sqrt{\sigma}\|_2^\dagger$.

Has a one-shot operational interpretation as the probability with which a purification of $\rho$ could pass a test for being a purification of $\sigma$.

Bures distance

Bures distance between $\rho$ and $\sigma$ is $D_B(\rho, \sigma) = \sqrt{2(1 - \sqrt{F(\rho, \sigma)})}$.

Mark M. Wilde (LSU)
Other measures of similarity for quantum states

**Trace distance**

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Other measures of similarity for quantum states

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**Fidelity [Uhl76]**

Fidelity between $\rho$ and $\sigma$ is $F(\rho, \sigma) \equiv \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$. Has a one-shot operational interpretation as the probability with which a purification of $\rho$ could pass a test for being a purification of $\sigma$. 
Other measures of similarity for quantum states

### Trace distance

Trace distance between $\rho$ and $\sigma$ is $\|\rho - \sigma\|_1$ where $\|A\|_1 = \text{Tr}\{\sqrt{A^\dagger A}\}$. Has a one-shot operational interpretation as the bias in success probability when distinguishing $\rho$ and $\sigma$ with an optimal quantum measurement.

### Fidelity [Uhl76]

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### Bures distance [Bur69]

Bures distance between $\rho$ and $\sigma$ is $D_B(\rho, \sigma) = \sqrt{2 \left(1 - \sqrt{F(\rho, \sigma)}\right)}$. 
Breakthrough result of [FR14]

Remainder term for strong subadditivity [FR14]

$\exists$ unitary channels $U_C$ and $V_{AC}$ such that

$$H(B|C) - \omega \geq -\log F(\omega_{ABC}, (V_{AC} \circ R_P C \to AC \circ U_C)(\omega_{BC}))$$

Nothing known from [FR14] about these unitaries! However, can conclude that $I(A;B|C)$ is small iff $\omega_{ABC}$ is approximately recoverable from system $C$ alone after the loss of system $A$.

Remainder term for monotonicity of relative entropy [BLW14]

$\exists$ unitary channels $U$ and $V$ such that

$$D(\rho \| \sigma) - D(N(\rho) \| N(\sigma)) \geq -\log F(\rho, (V \circ R_P \sigma, N \circ U)(N(\rho)))$$

Again, nothing known from [BLW14] about $U$ and $V$. 
Remainder term for strong subadditivity [FR14]

\[ H(B|C) - H(B|AC) \geq -\log F(\omega_{ABC}, (V_{AC} \circ R_{C \to AC} \circ U_{C})(\omega_{BC})) \]

Nothing known from [FR14] about these unitaries! However, can conclude that \( I(A; B|C) \) is small iff \( \omega_{ABC} \) is approximately recoverable from system \( C \) alone after the loss of system \( A \).
Breakthrough result of [FR14]

 Remainder term for strong subadditivity [FR14]

∃ unitary channels $\mathcal{U}_C$ and $\mathcal{V}_{AC}$ such that

$$H(B|C)_{\omega} - H(B|AC)_{\omega} \geq - \log F\left(\omega_{ABC}, (\mathcal{V}_{AC} \circ \mathcal{R}_C^{P}_{AC} \circ \mathcal{U}_C)(\omega_{BC})\right)$$

Nothing known from [FR14] about these unitaries! However, can conclude that $I(A; B|C)$ is small iff $\omega_{ABC}$ is approximately recoverable from system $C$ alone after the loss of system $A$.

 Remainder term for monotonicity of relative entropy [BLW14]

∃ unitary channels $\mathcal{U}$ and $\mathcal{V}$ such that

$$D(\rho||\sigma) - D(\mathcal{N}(\rho)||\mathcal{N}(\sigma)) \geq - \log F\left(\rho, (\mathcal{V} \circ \mathcal{R}_{\sigma,\mathcal{N}}^{P} \circ \mathcal{U})(\mathcal{N}(\rho))\right)$$

Again, nothing known from [BLW14] about $\mathcal{U}$ and $\mathcal{V}$. 
New result of [Wil15]

Let $\rho$ and $\sigma$ be such that $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and let $N$ be a quantum channel. Then the following inequality holds

$$D(\rho \parallel \sigma) - D(N(\rho) \parallel N(\sigma)) \geq -\log \left[ \sup_{t \in \mathbb{R}} F(\rho, R_P, t \sigma, N(N(\rho))) \right],$$

where $R_P, t \sigma, N$ is the following rotated Petz recovery map:

$$R_P, t \sigma, N(\cdot) \equiv (U_{\sigma, t} \circ R_P \sigma, N \circ U_N(\sigma), -t)(\cdot),$$

$R_P \sigma, N$ is the Petz recovery map, and $U_{\sigma, t}$ and $U_N(\sigma, -t)$ are defined from $U_\omega, t(\cdot) \equiv \omega \it(\cdot) - \it$, with $\omega$ a positive semi-definite operator.

Two tools for proof: Rényi generalization of a relative entropy difference and the Hadamard three-line theorem.
**New Theorem:** Let $\rho$ and $\sigma$ be such that $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and let $\mathcal{N}$ be a quantum channel. Then the following inequality holds

$$D(\rho \| \sigma) - D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)) \geq -\log \left[ \sup_{t \in \mathbb{R}} F \left( \rho, R_{\sigma, \mathcal{N}}^{P,t} (\mathcal{N}(\rho)) \right) \right],$$

where $R_{\sigma, \mathcal{N}}^{P,t}$ is the rotated Petz recovery map, and $U_{\sigma, t}$ and $U_{\mathcal{N}}(\sigma, -t)$ are defined from $U_{\omega, t}(\cdot) \equiv \omega \omega(\cdot) - t$, with $\omega$ a positive semi-definite operator.
New Theorem: Let $\rho$ and $\sigma$ be such that $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and let $\mathcal{N}$ be a quantum channel. Then the following inequality holds

$$D(\rho \parallel \sigma) - D(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma)) \geq -\log \left[ \sup_{t \in \mathbb{R}} F \left( \rho, \mathcal{R}^{P,t}_{\sigma,\mathcal{N}}(\mathcal{N}(\rho)) \right) \right],$$

where $\mathcal{R}^{P,t}_{\sigma,\mathcal{N}}$ is the following rotated Petz recovery map:

$$\mathcal{R}^{P,t}_{\sigma,\mathcal{N}}(\cdot) \equiv \left( \mathcal{U}_{\sigma,t} \circ \mathcal{R}^{P}_{\sigma,\mathcal{N}} \circ \mathcal{U}_{\mathcal{N}(\sigma),-t} \right)(\cdot),$$
**New Theorem:** Let $\rho$ and $\sigma$ be such that $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and let $\mathcal{N}$ be a quantum channel. Then the following inequality holds

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\log \left[ \sup_{t \in \mathbb{R}} F(\rho, \mathcal{R}_{\sigma,N}^{P,t}(\mathcal{N}(\rho))) \right],$$

where $\mathcal{R}_{\sigma,N}^{P,t}$ is the following rotated Petz recovery map:

$$\mathcal{R}_{\sigma,N}^{P,t}(\cdot) \equiv \left( \mathcal{U}_{\sigma,t} \circ \mathcal{R}_{\sigma,N}^{P} \circ \mathcal{U}_{\mathcal{N}(\sigma),-t} \right)(\cdot),$$

$\mathcal{R}_{\sigma,N}^{P}$ is the Petz recovery map, and $\mathcal{U}_{\sigma,t}$ and $\mathcal{U}_{\mathcal{N}(\sigma),-t}$ are defined from $\mathcal{U}_{\omega,t}(\cdot) \equiv \omega^{it}(\cdot)\omega^{-it}$, with $\omega$ a positive semi-definite operator.
**New Theorem:** Let $\rho$ and $\sigma$ be such that $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and let $\mathcal{N}$ be a quantum channel. Then the following inequality holds

$$D(\rho \parallel \sigma) - D(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma)) \geq -\log \left[ \sup_{t \in \mathbb{R}} F(\rho, \mathcal{R}_{\sigma,\mathcal{N}}^{P,t}(\mathcal{N}(\rho))) \right],$$

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$$\mathcal{R}_{\sigma,\mathcal{N}}^{P,t}(\cdot) \equiv (\mathcal{U}_{\sigma,t} \circ \mathcal{R}_{\sigma,\mathcal{N}}^{P} \circ \mathcal{U}_{\mathcal{N}(\sigma),-t})(\cdot),$$

$\mathcal{R}_{\sigma,\mathcal{N}}^{P}$ is the Petz recovery map, and $\mathcal{U}_{\sigma,t}$ and $\mathcal{U}_{\mathcal{N}(\sigma),-t}$ are defined from $\mathcal{U}_{\omega,t}(\cdot) \equiv \omega^{it}(\cdot)\omega^{-it}$, with $\omega$ a positive semi-definite operator.

**Two tools for proof:** Rényi generalization of a relative entropy difference and the Hadamard three-line theorem.
Rényi generalizations of a relative entropy difference

Definition from [BSW14, SBW14]

\[ \tilde{\Delta}^\alpha(\rho,\sigma, N) \equiv 2^{\alpha'} \log \| N(N(\rho)) - \alpha'/2 N(\sigma) \alpha'/2 \otimes I E \|_2 \]

where \( \alpha \in (0, 1) \cup (1, \infty) \), \( \alpha' \equiv (\alpha - 1)/\alpha \), and \( U_{S \to B}E \) is an isometric extension of \( N \).

Important properties

\[ \lim_{\alpha \to 1} \tilde{\Delta}^\alpha(\rho,\sigma, N) = D(\rho \| \sigma) - D(N(\rho) \| N(\sigma)). \]

\[ \tilde{\Delta}^{1/2}(\rho,\sigma, N) = -\log F(\rho, R_{\mathcal{P}}\sigma, N(N(\rho))). \]
Rényi generalizations of a relative entropy difference

Definition from [BSW14, SBW14]

\[
\tilde{\Delta}_\alpha (\rho, \sigma, \mathcal{N}) \equiv 2 \frac{\alpha'}{\alpha'} \log \left\| \left( \mathcal{N} (\rho)^{-\alpha'/2} \mathcal{N} (\sigma)^{\alpha'/2} \otimes I_E \right) U \sigma^{-\alpha'/2} \rho^{1/2} \right\|_{2\alpha},
\]

where \( \alpha \in (0, 1) \cup (1, \infty) \), \( \alpha' \equiv (\alpha - 1)/\alpha \), and \( U_{S \rightarrow BE} \) is an isometric extension of \( \mathcal{N} \).
Rényi generalizations of a relative entropy difference

Definition from [BSW14, SBW14]

\[ \tilde{\Delta}_\alpha (\rho, \sigma, \mathcal{N}) \equiv \frac{2}{\alpha'} \log \left\| \left( \mathcal{N} (\rho)^{-\alpha'/2} \mathcal{N} (\sigma)^{\alpha'/2} \otimes I_E \right) U \sigma^{-\alpha'/2} \rho^{1/2} \right\|_{2\alpha}, \]

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Important properties

\[ \lim_{\alpha \to 1} \tilde{\Delta}_\alpha (\rho, \sigma, \mathcal{N}) = D (\rho \| \sigma) - D (\mathcal{N} (\rho) \| \mathcal{N} (\sigma)). \]
\[ \tilde{\Delta}_{1/2} (\rho, \sigma, \mathcal{N}) = - \log F \left( \rho, \mathcal{R}^P_{\sigma, \mathcal{N}} (\mathcal{N} (\rho)) \right). \]
Hadamard three-line theorem

Let \( S = \{ z \in \mathbb{C} : 0 \leq \text{Re}\{z\} \leq 1 \} \), and let \( L(H) \) be the space of bounded linear operators acting on a Hilbert space \( H \). Let \( G : S \to L(H) \) be a bounded map that is holomorphic on the interior of \( S \) and continuous on the boundary. Let \( \theta \in (0,1) \) and define \( p_\theta \) by

\[
1 - \theta p_0 + \theta p_1,
\]

where \( p_0, p_1 \in [1, \infty] \). For \( k = 0, 1 \), define \( M_k = \sup_{t \in \mathbb{R}} \| G(k + it) \|_{p_k} \).

Then

\[
\| G(\theta) \|_{p_\theta} \leq M_1 - \theta M_0 + \theta M_1.
\]
Hadamard three-line theorem

Let \( S \equiv \{ z \in \mathbb{C} : 0 \leq \text{Re}\{z\} \leq 1 \} \), and let \( L(\mathcal{H}) \) be the space of bounded linear operators acting on a Hilbert space \( \mathcal{H} \). Let \( G : S \to L(\mathcal{H}) \) be a bounded map that is holomorphic on the interior of \( S \) and continuous on the boundary. Let \( \theta \in (0, 1) \) and define \( p_{\theta} \) by

\[
\frac{1}{p_{\theta}} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},
\]

where \( p_0, p_1 \in [1, \infty) \). For \( k = 0, 1 \), define \( M_k = \sup_{t \in \mathbb{R}} \| G(k + it) \|^{p_k} \). Then

\[
\| G(\theta) \|^{p_{\theta}} \leq M_1 - \theta M_0 + \theta M_1.
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Hadamard three-line theorem

Let $S \equiv \{ z \in \mathbb{C} : 0 \leq \text{Re}\{z\} \leq 1 \}$, and let $L(\mathcal{H})$ be the space of bounded linear operators acting on a Hilbert space $\mathcal{H}$. Let $G : S \to L(\mathcal{H})$ be a bounded map that is holomorphic on the interior of $S$ and continuous on the boundary. Let $\theta \in (0, 1)$ and define $p_\theta$ by

$$\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},$$

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$$M_k = \sup_{t \in \mathbb{R}} \| G(k + it) \|_{p_k}.$$
Let $S \equiv \{ z \in \mathbb{C} : 0 \leq \text{Re} \{z\} \leq 1 \}$, and let $L(\mathcal{H})$ be the space of bounded linear operators acting on a Hilbert space $\mathcal{H}$. Let $G : S \to L(\mathcal{H})$ be a bounded map that is holomorphic on the interior of $S$ and continuous on the boundary. Let $\theta \in (0, 1)$ and define $p_\theta$ by

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\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},
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$$
M_k = \sup_{t \in \mathbb{R}} \| G(k + it) \|_{p_k}.
$$

Then

$$
\| G(\theta) \|_{p_\theta} \leq M_0^{1-\theta} M_1^{\theta}.
$$
Three (or so) line proof

\[ G(z) \equiv \left( N(\rho) \right) z / 2 - \left( N(\sigma) \right) z / 2 \otimes I_E U \sigma z / 2 \rho 1 / 2, \]

\[ p_0 = 2, \quad p_1 = 1, \quad \theta \in (0, 1) \Rightarrow p_{\theta} = 2 \left( 1 + \theta \right) \]

\[ M_0 = \sup_{t \in \mathbb{R}} \| N(\rho) it / 2 - N(\sigma) it / 2 \otimes I_E U \sigma it \rho 1 / 2 \|_2 \leq \| \rho 1 / 2 \|_2 = 1, \]

\[ M_1 = \sup_{t \in \mathbb{R}} \| G(1 + it) \|_1 = \left[ \sup_{t \in \mathbb{R}} F(\rho, R_P, t \sigma, N(N(\rho))) \right]^{1/2}. \]

Apply the three-line theorem to conclude that

\[ \| G(\theta) \|_2 / (1 + \theta) \leq \left[ \sup_{t \in \mathbb{R}} F(\rho, R_P, t \sigma, N(N(\rho))) \right] \theta / 2. \]

Take a negative logarithm and the limit as \( \theta \downarrow 0 \) to conclude.
Three (or so) line proof

Pick $G(z) \equiv \left( [\mathcal{N}(\rho)]^{z/2} [\mathcal{N}(\sigma)]^{-z/2} \otimes I_E \right) U \sigma^{z/2} \rho^{1/2}$,

$p_0 = 2, \ p_1 = 1, \ \theta \in (0,1) \Rightarrow p_\theta = \frac{2}{1 + \theta}$
Three (or so) line proof

Pick $G(z) \equiv \left( [\mathcal{N}(\rho)]^{z/2} [\mathcal{N}(\sigma)]^{-z/2} \otimes I_E \right) U \sigma^{z/2} \rho^{1/2}$,

$p_0 = 2, \ p_1 = 1, \ \theta \in (0, 1) \Rightarrow p_\theta = \frac{2}{1 + \theta}$

$M_0 = \sup_{t \in \mathbb{R}} \left\| \left( \mathcal{N}(\rho)^{it/2} \mathcal{N}(\sigma)^{-it/2} \otimes I_E \right) U \sigma^{it} \rho^{1/2} \right\|_2 \leq \left\| \rho^{1/2} \right\|_2 = 1,$

$M_1 = \sup_{t \in \mathbb{R}} \| G(1 + it) \|_1 = \left[ \sup_{t \in \mathbb{R}} F \left( \rho, \mathcal{R}_{\sigma,t}^{P,t} (\mathcal{N}(\rho)) \right) \right]^{1/2}.$
Three (or so) line proof

Pick $G(z) \equiv \left( [\mathcal{N}(\rho)]^{z/2} [\mathcal{N}(\sigma)]^{-z/2} \otimes I_E \right) U \sigma^{z/2} \rho^{1/2},$

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$$M_0 = \sup_{t \in \mathbb{R}} \left\| \left( \mathcal{N}(\rho)^{it/2} \mathcal{N}(\sigma)^{-it/2} \otimes I_E \right) U \sigma^{it} \rho^{1/2} \right\|_2 \leq \left\| \rho^{1/2} \right\|_2 = 1,$$

$$M_1 = \sup_{t \in \mathbb{R}} \left\| G(1 + it) \right\|_1 = \left[ \sup_{t \in \mathbb{R}} F \left( \rho, \mathcal{R}_{\sigma,\mathcal{N}}^{P,t} (\mathcal{N}(\rho)) \right) \right]^{1/2}.$$

Apply the three-line theorem to conclude that

$$\left\| G(\theta) \right\|_{2/(1+\theta)} \leq \left[ \sup_{t \in \mathbb{R}} F \left( \rho, \mathcal{R}_{\sigma,\mathcal{N}}^{P,t} (\mathcal{N}(\rho)) \right) \right]^{\theta/2}.$$
Three (or so) line proof

Pick \( G(z) \equiv \left( [\mathcal{N} (\rho)]^{z/2} [\mathcal{N} (\sigma)]^{-z/2} \otimes I_E \right) U \sigma^{z/2} \rho^{1/2} \),

\[ p_0 = 2, \ p_1 = 1, \ \theta \in (0, 1) \Rightarrow p_\theta = \frac{2}{1 + \theta} \]

\[ M_0 = \sup_{t \in \mathbb{R}} \left\| \left( \mathcal{N} (\rho)^{it/2} \mathcal{N} (\sigma)^{-it/2} \otimes I_E \right) U \sigma^{it} \rho^{1/2} \right\|_2 \leq \left\| \rho^{1/2} \right\|_2 = 1, \]

\[ M_1 = \sup_{t \in \mathbb{R}} \| G(1 + it) \|_1 = \left[ \sup_{t \in \mathbb{R}} F \left( \rho, \mathcal{R}_{\sigma,\mathcal{N}}^{P,t} (\mathcal{N} (\rho)) \right) \right]^{1/2}. \]

Apply the three-line theorem to conclude that

\[ \| G(\theta) \|_{2/(1+\theta)} \leq \left[ \sup_{t \in \mathbb{R}} F \left( \rho, \mathcal{R}_{\sigma,\mathcal{N}}^{P,t} (\mathcal{N}(\rho)) \right) \right]^{\theta/2}. \]

Take a negative logarithm and the limit as \( \theta \searrow 0 \) to conclude.
Let $\rho_{ABC}$ be a density operator acting on a finite-dimensional Hilbert space $H_A \otimes H_B \otimes H_C$. Then the following inequality holds

$$I(A;B|C)_{\rho} \geq -\log\left[\sup_{t \in \mathbb{R}} F(\rho_{ABC}, R_P, t_{C \rightarrow AC}(\rho_{BC}))\right],$$

where $R_P, t_{C \rightarrow AC}$ is the following rotated Petz recovery map:

$$R_P, t_{C \rightarrow AC}(\cdot) \equiv (U_{\rho_{AC}}, t \circ R_P_{C \rightarrow AC} \circ U_{\rho_C}, -t)(\cdot),$$

the Petz recovery map $R_P_{C \rightarrow AC}$ is defined as

$$R_P_{C \rightarrow AC}(\cdot) \equiv \rho_{1/2 AC}\left[\rho - \frac{1}{2}C(\cdot)\rho - \frac{1}{2}C\otimes I_A\right]\rho_{1/2 AC},$$

and the partial isometric maps $U_{\rho_{AC}}, t$ and $U_{\rho_C}, -t$ are defined as before.
Let $\rho_{ABC}$ be a density operator acting on a finite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Then the following inequality holds

$$I(A; B|C)_{\rho} \geq - \log \left[ \sup_{t \in \mathbb{R}} F \left( \rho_{ABC}, \mathcal{R}_{C \rightarrow AC}^{P,t} (\rho_{BC}) \right) \right],$$

where $\mathcal{R}_{C \rightarrow AC}^{P,t}$ is the following rotated Petz recovery map:

$$\mathcal{R}_{C \rightarrow AC}^{P,t} (\cdot) \equiv (U_{\rho_{AC}}(t) \circ \mathcal{R}_{C \rightarrow AC} \circ U_{\rho_{C}}(-t))(\cdot),$$

the Petz recovery map $\mathcal{R}_{C \rightarrow AC}$ is defined as

$$\mathcal{R}_{C \rightarrow AC} (\cdot) \equiv \rho_{1}/2_{AC} \left[ \rho - \rho_{C} \otimes I_A \right] \rho_{1}/2_{AC},$$

and the partial isometric maps $U_{\rho_{AC}}$ and $U_{\rho_{C}}$ are defined as before.
SSA refinement as a special case

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where $\mathcal{R}^{P,t}_{C \rightarrow AC}$ is the following rotated Petz recovery map:

$$\mathcal{R}^{P,t}_{C \rightarrow AC} (\cdot) \equiv \left( \mathcal{U}_{\rho_{AC}, t} \circ \mathcal{R}^{P}_{C \rightarrow AC} \circ \mathcal{U}_{\rho_{C}, -t} \right) (\cdot),$$

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the Petz recovery map $\mathcal{R}^{P}_{C \to AC}$ is defined as

$$\mathcal{R}^{P}_{C \to AC}(\cdot) \equiv \rho_{AC}^{1/2} \left[ \rho_{C}^{-1/2}(\cdot) \rho_{C}^{-1/2} \otimes I_A \right] \rho_{AC}^{1/2},$$
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Conclusions

The result of [FR14] already had a number of important implications in quantum information theory. The new result in [Wil15] applies to relative entropy differences, has a brief proof, and improves our understanding of the input and output unitaries (but see [SFR15] for the special case of SSA). By building on [SFR15, Wil15], we can now generalize these results: there is a universal recovery map which depends only on $\sigma$ and $N$ and has the form [SRWW15]:

$$X \rightarrow \int \mu(\text{d}t) R_{P, \sigma, N}(X)$$

for some probability measure $\mu$. It is still conjectured that the recovery map can be the Petz recovery map alone (not a rotated Petz map).
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