Recoverability in quantum information theory

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- They are helpful in determining the ultimate limits on many physical processes (communication, thermodynamics, uncertainty relations)
- Many of these entropy inequalities are equivalent to each other, so we can say that together they constitute a fundamental law of quantum information theory
- There has been recent interest in refining these inequalities, trying to understand how well one can attempt to reverse an irreversible physical process
- This talk discusses progress in this direction

Background — entropies

Umegaki relative entropy [Ume62]

The quantum relative entropy is a measure of dissimilarity between two quantum states. Defined for state ρ and positive semi-definite σ as

$$D(\rho\|\sigma) \equiv \mathrm{Tr}\{\rho[\log \rho - \log \sigma]\}$$

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Physical interpretation with quantum Stein's lemma [HP91, NO00]

Given are n quantum systems, all of which are prepared in either the state ρ or σ. With a constraint of $\varepsilon \in (0,1)$ on the Type I error of misidentifying ρ, then the optimal error exponent for the Type II error of misidentifying σ is $D(\rho\|\sigma)$.

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$$D(\rho\|\sigma) \geq D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

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- quantum Stein's lemma [BS03]

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- Conditional entropy does not decrease under the loss of system A:

$$H(B|C)_\rho \geq H(B|AC)_\rho$$

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- This "Petz" recovery map has the following explicit form [HJPW04]:

$$\mathcal{R}^P_{\sigma,\mathcal{N}}(\omega) \equiv \sigma^{1/2}\mathcal{N}^\dagger\left((\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right)\sigma^{1/2}$$

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- Classical case: Distributions p_X and q_X and a channel $\mathcal{N}(y|x)$. Then the Petz recovery map $\mathcal{R}^P(x|y)$ is given by the Bayes theorem:

$$\mathcal{R}^P(x|y)q_Y(y) = \mathcal{N}(y|x)q_X(x)$$

where $q_Y(y) \equiv \sum_x \mathcal{N}(y|x)q_X(x)$

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- Linear, completely positive by inspection and trace non-increasing because

$$\begin{aligned}\text{Tr}\{\mathcal{R}^{P}_{\sigma,\mathcal{N}}(\omega)\} &= \text{Tr}\{\sigma^{1/2}\mathcal{N}^{\dagger}\left((\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right)\sigma^{1/2}\} \\ &= \text{Tr}\{\sigma\mathcal{N}^{\dagger}\left((\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\right)\} \\ &= \text{Tr}\{\mathcal{N}(\sigma)(\mathcal{N}(\sigma))^{-1/2}\omega(\mathcal{N}(\sigma))^{-1/2}\} \\ &\leq \text{Tr}\{\omega\}\end{aligned}$$

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$$

- For $\mathcal{N}(\sigma)$ positive definite, the map perfectly recovers σ from $\mathcal{N}(\sigma)$:

$$
\begin{aligned}
\mathcal{R}^{P}_{\sigma,\mathcal{N}}(\mathcal{N}(\sigma)) &= \sigma^{1/2}\mathcal{N}^{\dagger}\left((\mathcal{N}(\sigma))^{-1/2}\mathcal{N}(\sigma)(\mathcal{N}(\sigma))^{-1/2}\right)\sigma^{1/2} \\
&= \sigma^{1/2}\mathcal{N}^{\dagger}(I)\,\sigma^{1/2} \\
&= \sigma
\end{aligned}
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Tensorial [LW14]

Given a tensor-product state and channel, then the Petz recovery map is a tensor product: $\mathcal{R}^{P}_{\sigma_1\otimes\sigma_2,\mathcal{N}_1\otimes\mathcal{N}_2} = \mathcal{R}^{P}_{\sigma_1,\mathcal{N}_1} \otimes \mathcal{R}^{P}_{\sigma_2,\mathcal{N}_2}$. "Individual action suffices for 'pretty good' recovery of individual states"

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Composition [LW14]

Given $\mathcal{N}_2 \circ \mathcal{N}_1$, then $\mathcal{R}^{P}_{\sigma,\mathcal{N}_2\circ\mathcal{N}_1} = \mathcal{R}^{P}_{\sigma,\mathcal{N}_1} \circ \mathcal{R}^{P}_{\mathcal{N}_1(\sigma),\mathcal{N}_2}$. "To recover 'pretty well' overall, recover 'pretty well' from the last noise first and the first noise last"

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- Then $\mathcal{N}^\dagger(\cdot) = (\cdot) \otimes I_A$ and Petz recovery map is

$$\mathcal{R}^P_{C \to AC}(\tau_C) = \omega_{AC}^{1/2} \left(\omega_C^{-1/2} \tau_C \omega_C^{-1/2} \otimes I_A \right) \omega_{AC}^{1/2}$$

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- Interpretation: If system A is lost but $H(B|C)_\omega = H(B|AC)_\omega$, then one can recover the full state on ABC by performing the Petz recovery map on system C of ω_{BC}, i.e.,

$$\omega_{ABC} = \mathcal{R}^P_{C\to AC}(\omega_{BC})$$

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Approximate case for strong subadditivity

- What can we say when $H(B|C)_\omega - H(B|AC)_\omega = \varepsilon$?
- Is ω_{ABC} approximately recoverable from ω_{BC} by performing a recovery map on system C alone? [WL12]

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Trace distance

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Fidelity [Uhl76]

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Bures distance [Bur69]

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Remainder term for strong subadditivity [FR14]

$\exists$ unitary channels $\mathcal{U}_C$ and $\mathcal{V}_{AC}$ such that

$$H(B|C)_\omega - H(B|AC)_\omega \geq -\log F\left(\omega_{ABC}, (\mathcal{V}_{AC} \circ \mathcal{R}^P_{C\to AC} \circ \mathcal{U}_C)(\omega_{BC})\right)$$

Nothing known from [FR14] about these unitaries! However, can conclude that $I(A;B|C)$ is small iff ω_{ABC} is approximately recoverable from system C alone after the loss of system A.

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Remainder term for monotonicity of relative entropy [BLW14]

$\exists$ unitary channels $\mathcal{U}$ and $\mathcal{V}$ such that

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\log F\Big(\rho, (\mathcal{V} \circ \mathcal{R}^P_{\sigma,\mathcal{N}} \circ \mathcal{U})(\mathcal{N}(\rho))\Big)$$

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$$\mathcal{R}^{P,t}_{\sigma,\mathcal{N}}(\cdot) \equiv \left(\mathcal{U}_{\sigma,t} \circ \mathcal{R}^{P}_{\sigma,\mathcal{N}} \circ \mathcal{U}_{\mathcal{N}(\sigma),-t}\right)(\cdot),$$

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$\mathcal{R}^{P}_{\sigma,\mathcal{N}}$ is the Petz recovery map, and $\mathcal{U}_{\sigma,t}$ and $\mathcal{U}_{\mathcal{N}(\sigma),-t}$ are defined from $\mathcal{U}_{\omega,t}(\cdot) \equiv \omega^{it}(\cdot)\omega^{-it}$, with ω a positive semi-definite operator.

New result of [Wil15]

New Theorem: Let ρ and σ be such that $\text{supp}(\rho) \subseteq \text{supp}(\sigma)$ and let $\mathcal{N}$ be a quantum channel. Then the following inequality holds

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq -\log\left[\sup_{t\in\mathbb{R}} F\left(\rho, \mathcal{R}^{P,t}_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))\right)\right],$$

where $\mathcal{R}^{P,t}_{\sigma,\mathcal{N}}$ is the following rotated Petz recovery map:

$$\mathcal{R}^{P,t}_{\sigma,\mathcal{N}}(\cdot) \equiv \left(\mathcal{U}_{\sigma,t} \circ \mathcal{R}^{P}_{\sigma,\mathcal{N}} \circ \mathcal{U}_{\mathcal{N}(\sigma),-t}\right)(\cdot),$$

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Two tools for proof: Rényi generalization of a relative entropy difference and the Hadamard three-line theorem

Rényi generalizations of a relative entropy difference

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Definition from [BSW14, SBW14]

$$\widetilde{\Delta}_\alpha(\rho, \sigma, \mathcal{N}) \equiv \frac{2}{\alpha'} \log \left\| \left(\mathcal{N}(\rho)^{-\alpha'/2} \mathcal{N}(\sigma)^{\alpha'/2} \otimes I_E \right) U \sigma^{-\alpha'/2} \rho^{1/2} \right\|_{2\alpha},$$

where $\alpha \in (0,1) \cup (1, \infty)$, $\alpha' \equiv (\alpha - 1)/\alpha$, and $U_{S \to BE}$ is an isometric extension of $\mathcal{N}$.

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Important properties

$$\lim_{\alpha\to 1} \widetilde{\Delta}_\alpha(\rho,\sigma,\mathcal{N}) = D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)).$$

$$\widetilde{\Delta}_{1/2}(\rho,\sigma,\mathcal{N}) = -\log F\left(\rho, \mathcal{R}^P_{\sigma,\mathcal{N}}(\mathcal{N}(\rho))\right).$$

Hadamard three-line theorem

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Let $S \equiv \{z \in \mathbb{C} : 0 \leq \operatorname{Re}\{z\} \leq 1\}$, and let $L(\mathcal{H})$ be the space of bounded linear operators acting on a Hilbert space $\mathcal{H}$. Let $G : S \to L(\mathcal{H})$ be a bounded map that is holomorphic on the interior of S and continuous on the boundary. Let $\theta \in (0,1)$ and define p_θ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

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$$M_k = \sup_{t \in \mathbb{R}} \|G(k + it)\|_{p_k}.$$

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Then

$$\left\| G(\theta) \right\|_{p_\theta} \leq M_0^{1-\theta} M_1^{\theta}.$$

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$$\text{Pick } G(z) \equiv \left([\mathcal{N}(\rho)]^{z/2} [\mathcal{N}(\sigma)]^{-z/2} \otimes I_E \right) U \sigma^{z/2} \rho^{1/2},$$

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$$M_1 = \sup_{t \in \mathbb{R}} \| G(1+it) \|_1 = \left[\sup_{t \in \mathbb{R}} F\left(\rho, \mathcal{R}^{P,t}_{\sigma,\mathcal{N}}(\mathcal{N}(\rho)) \right) \right]^{1/2}.$$

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Apply the three-line theorem to conclude that

$$\| G(\theta) \|_{2/(1+\theta)} \leq \left[\sup_{t\in\mathbb{R}} F\left(\rho, \mathcal{R}^{P,t}_{\sigma,\mathcal{N}}(\mathcal{N}(\rho)) \right) \right]^{\theta/2}.$$

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Take a negative logarithm and the limit as $\theta \searrow 0$ to conclude.

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Let ρ_{ABC} be a density operator acting on a finite-dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Then the following inequality holds

$$I(A;B|C)_\rho \geq -\log\left[\sup_{t\in\mathbb{R}} F\left(\rho_{ABC}, \mathcal{R}^{P,t}_{C\to AC}(\rho_{BC})\right)\right],$$

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and the partial isometric maps $\mathcal{U}_{\rho_{AC},t}$ and $\mathcal{U}_{\rho_C,-t}$ are defined as before.

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- By building on [SFR15, Wil15], we can now generalize these results: there is a universal recovery map which depends only on σ and $\mathcal{N}$ and has the form [SRWW15]:

$$X \to \int \mu(dt)\, \mathcal{R}^{P,t}_{\sigma,\mathcal{N}}(X)$$

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- It is still conjectured that the recovery map can be the Petz recovery map alone (not a rotated Petz map).

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