Converse bounds for private communication over quantum channels

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Main message

- We establish a general meta-converse bound for private communication protocols, by using the notion of a private state and a "privacy" test to determine whether a given state is a private state
- The meta-converse bound has a number of applications, including strong converse bounds and second-order characterizations of private communication
- The bounds are related to the relative entropy of entanglement and sharpen known upper bounds on rates of quantum key distribution protocols
- We establish the strong converse property for the two-way assisted private capacity of the pure-loss and quantum-limited amplifier channels. We also get strong converse rates for other quantum Gaussian channels.

Private states

Tripartite key state

A tripartite key state γ_{ABE} contains $\log K$ bits of secret key if there exists a state σ_E such that

$$\gamma_{ABE} = \frac{1}{K} \sum_i |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B \otimes \sigma_E.$$

Bipartite private state

A bipartite private state $\gamma_{ABA'B'}$ has the following form:

$$\gamma_{ABA'B'} = U_{ABA'B'}(\Phi_{AB} \otimes \theta_{A'B'})U^\dagger_{ABA'B'},$$

where $U_{ABA'B'}$ is a "twisting" unitary of the form $U_{ABA'B'} = \sum_{i,j} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B \otimes U^{ij}_{A'B'}$, with each $U^{ij}_{A'B'}$ a unitary, and $\theta_{A'B'}$ a state.

Private states

- The systems A' and B' are called the "shield" systems because they, along with the twisting unitary, can help to protect the key in systems AB from any party possessing a purification of $\gamma_{ABA'B'}$.

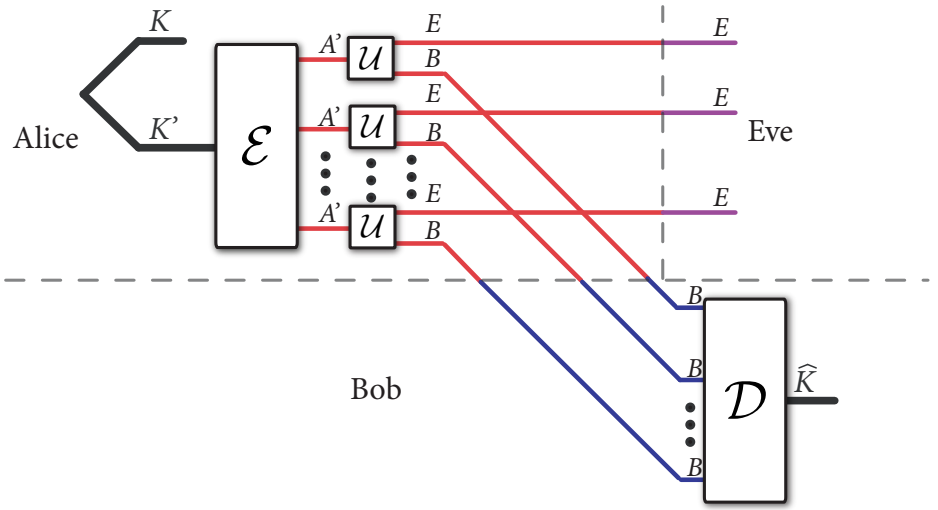
- Such bipartite private states are in one-to-one correspondence with tripartite key states. That is, for every tripartite key state γ_{ABE}, we can find a bipartite private state and vice versa.

- This correspondence takes on a more physical form: any tripartite protocol whose aim it is to extract tripartite key states is in 1-to-1 correspondence with a bipartite protocol whose aim it is to extract bipartite private states.

Private communication protocols

Unassisted private communication

- Given is a quantum channel $\mathcal{N}_{A' \to B}$. Let $U^{\mathcal{N}}_{A' \to BE}$ be an isometric extension of $\mathcal{N}_{A' \to B}$.
- A secret-key generation protocol for n channel uses consists of a triple $\{|K|, \mathcal{E}, \mathcal{D}\}$, where $|K|$ is the size of the secret key to be generated, $\mathcal{E}_{K' \to A'^n}$ is the encoder, and $\mathcal{D}_{B^n \to \hat{K}}$ is the decoder.



Private communication protocols

Unassisted private communication

- A triple (n, P, ε) consists of the number n of channel uses, the rate P of secret-key generation, and the error $\varepsilon \in [0, 1]$.
- Such a triple is achievable on $\mathcal{N}_{A' \to B}$ if there exists a secret-key generation protocol $\{|K|, \mathcal{E}, \mathcal{D}\}$ and some state ω_{E^n} such that $\frac{1}{n} \log |K| \geq P$ and

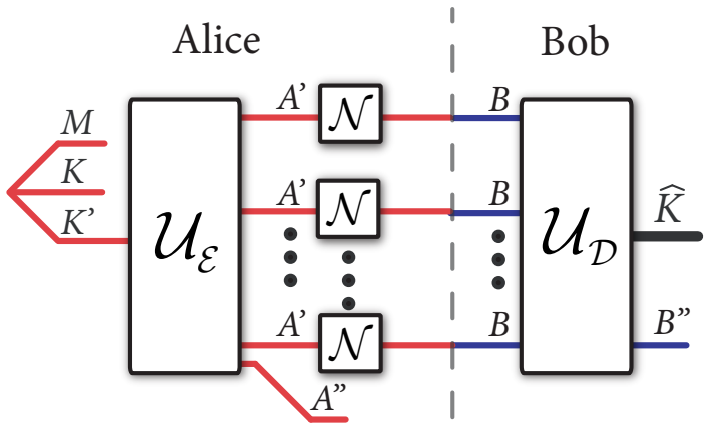
$$F(\overline{\Phi}_{K\hat{K}} \otimes \omega_{E^n}, \rho_{K\hat{K}E^n}) \geq 1 - \varepsilon,$$

where $\rho_{K\hat{K}E^n} \equiv (\mathcal{D}_{B^n \to \hat{K}} \circ (\mathcal{U}^{\mathcal{N}}_{A' \to BE})^{\otimes n} \circ \mathcal{E}_{K' \to A'^n})(\overline{\Phi}_{KK'})$ and

$$\overline{\Phi}_{KK'} \equiv \frac{1}{|K|} \sum_{i=0}^{|K|-1} |i\rangle\langle i|_K \otimes |i\rangle\langle i|_{K'}.$$

Equivalent bipartite protocol

Can reformulate such a protocol in the bipartite picture: perform every step coherently, with the goal to produce a bipartite private state



Due to equivalence between tripartite and bipartite pictures

$$F(\gamma_{K_A K_B S_A S_B}, \rho_{K\hat{K}MA''B''}) \geq 1 - \varepsilon,$$

for some private state $\gamma_{K_A K_B S_A S_B}$, where we identify $K_A \equiv K$, $K_B \equiv \hat{K}$, $S_A \equiv MA''$, and $S_B \equiv B''$, and

$$\rho_{K\hat{K}MA''B''} \equiv (\mathcal{U}^{\mathcal{D}}_{B^n \to \hat{K}B''} \circ (\mathcal{U}^{\mathcal{N}}_{A' \to BE})^{\otimes n} \circ \mathcal{U}^{\mathcal{E}}_{K' \to A'^n A''})(\Phi^{\mathrm{GHZ}}_{KK'M}).$$

Non-asymptotic achievable region

Non-asymptotic fundamental limits

Boundaries of the achievable regions:

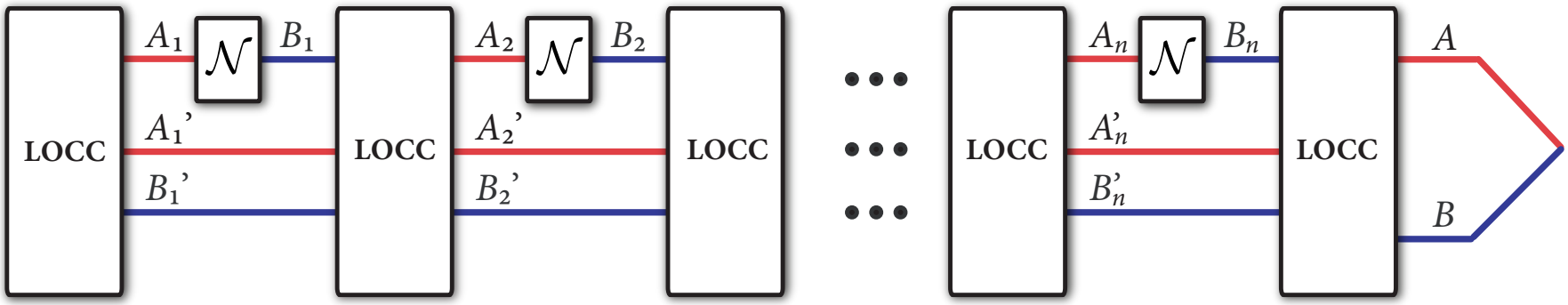
$$\hat{P}_{\mathcal{N}}(n,\varepsilon) \equiv \max\{P : (n,P,\varepsilon) \text{ is achievable for } \mathcal{N}\},$$
$$\hat{\varepsilon}_{\mathcal{N}}(n,P) \equiv \min\{\varepsilon : (n,P,\varepsilon) \text{ is achievable for } \mathcal{N}\}.$$

Interpretations

- 1st boundary $\hat{P}_{\mathcal{N}}(n,\varepsilon)$ identifies how rate can change as a function of n for fixed error ε, and 2nd-order coding rates can characterize it
- 2nd boundary $\hat{\varepsilon}_{\mathcal{N}}(n,P)$ identifies how error can change as a function of n for fixed rate R, and error exponents and strong converse exponents characterize it

LOCC-assisted private communication protocols

LOCC-assisted protocols are defined similarly, but allow for rounds of LOCC between channel uses (like in QKD)



Define boundaries of non-asymptotic achievable regions similarly as

$$\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon) \equiv \max\{P : (n,P,\varepsilon) \text{ is achievable for } \mathcal{N} \text{ using } \leftrightarrow\},$$
$$\hat{\varepsilon}^{\leftrightarrow}_{\mathcal{N}}(n,P) \equiv \min\{\varepsilon : (n,P,\varepsilon) \text{ is achievable for } \mathcal{N} \text{ using } \leftrightarrow\}.$$

Information measures

- Hypothesis testing relative entropy defined for a state ρ, positive semi-definite operator σ, and $\varepsilon \in [0,1]$ as

$$D_H^\varepsilon(\rho\|\sigma) \equiv -\log\left[\min\{\mathrm{Tr}\{\Lambda\sigma\} : 0 \leq \Lambda \leq I \wedge \mathrm{Tr}\{\Lambda\rho\} \geq 1-\varepsilon\}\right].$$

- Has a second-order expansion for i.i.d. states:

$$D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = nD(\rho\|\sigma) + \sqrt{nV(\rho\|\sigma)}\Phi^{-1}(\varepsilon) + O(\log n).$$

where
$$D(\rho\|\sigma) \equiv \mathrm{Tr}\{\rho[\log\rho - \log\sigma]\},$$
$$V(\rho\|\sigma) \equiv \mathrm{Tr}\{\rho[\log\rho - \log\sigma - D(\rho\|\sigma)]^2\}$$
$$\Phi(a) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} dx\ \exp\left(-x^2/2\right)$$

Information measures

- Can use hypothesis testing relative entropy to define the ε-relative entropy of entanglement:

$$E_R^{\varepsilon}(A;B)_\rho \equiv \inf_{\sigma_{AB} \in \mathcal{S}(A:B)} D_H^{\varepsilon}(\rho_{AB} \| \sigma_{AB}).$$

  where $\mathcal{S}(A:B)$ is the set of separable states

- Can also define a channel's ε-relative entropy of entanglement:

$$E_R^{\varepsilon}(\mathcal{N}) \equiv \sup_{|\psi\rangle_{AA'} \in \mathcal{H}_{AA'}} E_R^{\varepsilon}(A;B)_\rho,$$

  where $\rho_{AB} \equiv \mathcal{N}_{A' \to B}(\psi_{AA'})$

- Standard relative entropies of entanglement defined by replacing D_H^{ε} with quantum relative entropy D

Privacy test

- Can test whether a given state is a γ-private state by "untwisting" and projecting onto the maximally entangled state:

$$\{\Pi_{ABA'B'}, I_{ABA'B'} - \Pi_{ABA'B'}\},$$

where $\Pi_{ABA'B'} \equiv U_{ABA'B'}\left(\Phi_{AB} \otimes I_{A'B'}\right) U^{\dagger}_{ABA'B'}$.

- Let $\varepsilon \in [0,1]$ and let $\rho_{ABA'B'}$ be an ε-approximate γ-private state. The probability for $\rho_{ABA'B'}$ to pass the γ-privacy test satisfies

$$\mathrm{Tr}\{\Pi_{ABA'B'}\rho_{ABA'B'}\} \geq 1 - \varepsilon,$$

- For a separable state $\sigma_{ABA'B'} \in \mathcal{S}(AA' : BB')$, the probability of passing any γ-privacy test is never larger than $1/K$:

$$\mathrm{Tr}\{\Pi_{ABA'B'}\sigma_{ABA'B'}\} \leq \frac{1}{K},$$

where K is the number of values that the secret key can take.

Meta-converse bound for private communication

Theorem

For any fixed $\varepsilon \in (0,1)$, the achievable region satisfies

$$\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N}).$$

The same bound holds when allowing for a round of LOCC before and after the channel use.

Proof idea: use monotonicity of E_R^{ε} with respect to LOCC and use the bounds on the previous slide.

Meta-converse bound for private communication

Corollary

The following bound holds for n channel uses:

$$\hat{P}_{\mathcal{N}}(n,\varepsilon) \leq \frac{1}{n} E_R^{\varepsilon}(\mathcal{N}^{\otimes n}).$$

The same bound holds when allowing for rounds of LOCC before and after all n channel uses. The same bound holds for $\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon)$ *if the channel* $\mathcal{N}$ *is teleportation simulable.*

Application: Strong converse rate for private capacity

Theorem

For any channel $\mathcal{N}$, its relative entropy of entanglement is a strong converse rate for private communication:

$$P^{\dagger}(\mathcal{N}) \leq E_R(\mathcal{N}),$$

where $P^{\dagger}(\mathcal{N})$ is the strong converse private capacity of $\mathcal{N}$.

Proof idea: Use permutation symmetry of an i.i.d. channel, de Finetti theorem, and a Rényi version of the relative entropy of entanglement.

Theorem

If $\mathcal{N}$ is teleportation simulable, then its relative entropy of entanglement is an upper bound for the LOCC-assisted strong converse private capacity:

$$P^{\dagger}_{\leftrightarrow}(\mathcal{N}) \leq E_R(\mathcal{N}).$$

Application: Second-order expansions of converse bounds

Theorem

If a quantum channel $\mathcal{N}_{A'\to B}$ is covariant, then

$$\hat{P}_{\mathcal{N}}(n,\varepsilon) \leq E_R(A;B)_\rho + \sqrt{\frac{V^{\varepsilon}_{E_R}(A;B)_\rho}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right),$$

where $\rho_{AB} = \mathcal{N}_{A'\to B}(\Phi_{AA'})$. If a quantum channel $\mathcal{N}_{A'\to B}$ is teleportation-simulable with associated state ω_{AB}, then

$$\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon) \leq E_R(A;B)_\omega + \sqrt{\frac{V^{\varepsilon}_{E_R}(A;B)_\omega}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right).$$

where $$V^{\varepsilon}_{E_R}(A;B)_\rho \equiv \begin{cases} \max_{\sigma_{AB'}\in\Pi_{\mathcal{S}}} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon < 1/2 \\ \min_{\sigma_{AB}\in\Pi_{\mathcal{S}}} V(\rho_{AB}\|\sigma_{AB}) & \text{for } \varepsilon \geq 1/2 \end{cases},$$

with $\Pi_{\mathcal{S}} \subseteq \mathcal{S}(A\!:\!B)$ the set of states achieving minimum in $E_R(A;B)_\rho$

Example: Dephasing channel

Theorem

For the qubit dephasing channel $\mathcal{Z}^{\gamma} : \rho \mapsto (1-\gamma)\,\rho + \gamma Z \rho Z$, *with* $\gamma \in (0,1)$, *the boundary* $\hat{P}(n;\varepsilon)$ *satisfies*

$$\hat{P}(n,\varepsilon) = \hat{P}^{\leftrightarrow}(n,\varepsilon) = 1 - h(\gamma) + \sqrt{\frac{v(\gamma)}{n}}\,\Phi^{-1}(\varepsilon) + \frac{\log n}{2n} + O\left(\frac{1}{n}\right),$$

where Φ *is the cumulative standard Gaussian distribution,* $h(\gamma)$ *denotes the binary entropy and* $v(\gamma)$ *the corresponding variance, defined as*

$$h(\gamma) \equiv -\gamma \log \gamma - (1-\gamma)\log(1-\gamma),$$
$$v(\gamma) \equiv \gamma(\log\gamma + h(\gamma))^2 + (1-\gamma)(\log(1-\gamma) + h(\gamma))^2.$$

Example: Erasure channel

Theorem

For the qubit erasure channel $\mathcal{E}^p_{A'\to B} : \rho_{A'} \mapsto (1-p)\rho_B + p|e\rangle\langle e|_B$ *with* $p \in (0,1)$, *the boundary* $\hat{P}^{\leftrightarrow}_{\mathcal{E}^p}(n,\varepsilon)$ *satisfies*

$$\varepsilon = \sum_{l=n-k+1}^{n} \binom{n}{l} p^l (1-p)^{n-l} \left(1 - 2^{n\left(1-\hat{P}^{\leftrightarrow}_{\mathcal{E}^p}(n,\varepsilon)\right)-l}\right) .$$

Moreover, the following expansion holds

$$\hat{P}^{\leftrightarrow}_{\mathcal{E}^p}(n,\varepsilon) = 1 - p + \sqrt{\frac{p(1-p)}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{1}{n}\right) .$$

Application: Quantum Gaussian channels

Definitions of quantum Gaussian channels

$$\text{Thermal channel } \mathcal{L}_{\eta,N_B}: \quad \hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e},$$
$$\text{Amplifier channel } \mathcal{A}_{G,N_B}: \quad \hat{b} = \sqrt{G}\hat{a} + \sqrt{G-1}\hat{e}^\dagger,$$
$$\text{Additive-noise channel } \mathcal{W}_\xi: \quad \hat{b} = \hat{a} + (x+ip)/\sqrt{2},$$

- Thermal channel has transmissivity $\eta \in [0,1]$ and environment prepared in thermal state of mean photon number N_B.
- Amplifier channel has gain $G \in [1,\infty)$ and environment prepared in thermal state of mean photon number N_B. If $N_B = 0$, then channels are quantum-limited.
- Additive noise channel has x and p be zero-mean Gaussian random variables with variance $\xi \geq 0$.

Unconstrained relative entropy variances

Let $V_{\mathcal{L}_{\eta,N_B}}$, $V_{\mathcal{A}_{G,N_B}}$, and $V_{\mathcal{W}_\xi}$ be the unconstrained relative entropy variances of the thermalizing, amplifier, and additive-noise channels, respectively:

$$\begin{aligned} V_{\mathcal{L}_{\eta,N_B}} &\equiv N_B(N_B+1)\log^2(\eta\left[N_B+1\right]/N_B), \\ V_{\mathcal{A}_{G,N_B}} &\equiv N_B(N_B+1)\log^2(G^{-1}\left[N_B+1\right]/N_B), \\ V_{\mathcal{W}_\xi} &\equiv (1-\xi)^2/\ln^2 2. \end{aligned}$$

Application: Strong converse rates for Gaussian channels

Theorem

The following converse bounds hold for $\varepsilon \in (0,1)$*:*

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_{\eta,N_B}}(n,\varepsilon) \leq -\log\left((1-\eta)\,\eta^{N_B}\right) - g(N_B) + \sqrt{\frac{2V_{\mathcal{L}_{\eta,N_B}}}{n(1-\varepsilon)}} + O(1/n),$$

$$\hat{P}^{\leftrightarrow}_{\mathcal{A}_{G,N_B}}(n,\varepsilon) \leq \log\left(\frac{G^{N_B+1}}{G-1}\right) - g(N_B) + \sqrt{\frac{2V_{\mathcal{A}_{G,N_B}}}{n(1-\varepsilon)}} + O(1/n),$$

$$\hat{P}^{\leftrightarrow}_{\mathcal{W}_\xi}(n,\varepsilon) \leq \frac{\xi-1}{\ln 2} - \log\xi + \sqrt{\frac{2V_{\mathcal{W}_\xi}}{n(1-\varepsilon)}} + O(1/n).$$

Proof idea: Use a teleportation simulation argument, employ meta-converse theorem, apply Chebyshev inequality, a formula for the relative entropy variance of two Gaussian states, and take the infinite-energy limit

Application: Strong converse for quantum-limited channels

Corollary

For the pure-loss channel $\mathcal{L}_\eta$ and quantum-limited amplifier channel $\mathcal{A}_G$, the following bounds hold

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq -\log(1-\eta) + O\left(\frac{1}{n}\right),$$

$$\hat{P}^{\leftrightarrow}_{\mathcal{A}_G}(n,\varepsilon) \leq \log\left(\frac{G}{G-1}\right) + O\left(\frac{1}{n}\right).$$

Tool: Relative entropy variance for Gaussian states

Writing zero-mean Gaussian states in exponential form as

$$\rho = Z_\rho^{-1/2} \exp\left\{-\frac{1}{2}\hat{x}^T G_\rho \hat{x}\right\}, \qquad \sigma = Z_\sigma^{-1/2} \exp\left\{-\frac{1}{2}\hat{x}^T G_\sigma \hat{x}\right\},$$

where

$$Z_\rho \equiv \det(V^\rho + i\Omega/2), \qquad Z_\sigma \equiv \det(V^\sigma + i\Omega/2),$$
$$G_\rho \equiv 2i\Omega \operatorname{arcoth}(2V^\rho i\Omega), \qquad G_\sigma \equiv 2i\Omega \operatorname{arcoth}(2V^\sigma i\Omega),$$

and V^ρ and V^σ are Wigner function covariance matrices for ρ and σ.

Theorem

For zero-mean Gaussian states ρ and σ, the relative entropy variance is

$$V(\rho\|\sigma) = \frac{1}{2}\operatorname{Tr}\{\Delta V^\rho \Delta V^\rho\} + \frac{1}{8}\operatorname{Tr}\{\Delta\Omega\Delta\Omega\},$$

where $\Delta \equiv G_\rho - G_\sigma$.