

# Can classical noise enhance quantum transmission?

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## Abstract

A modified quantum teleportation protocol broadens the scope of the classical forbidden-interval theorems for stochastic resonance. The fidelity measures performance of quantum communication. The sender encodes the two classical bits for quantum teleportation as weak bipolar subthreshold signals and sends them over a noisy classical channel. Two forbidden-interval theorems provide a necessary and sufficient condition for the occurrence of the nonmonotone stochastic resonance effect in the fidelity of quantum teleportation. The condition is that the noise mean must fall outside a forbidden interval related to the detection threshold and signal value. An optimal amount of classical noise benefits quantum communication when the sender transmits weak signals, the receiver detects with a high threshold and the noise mean lies outside the forbidden interval. Theorems and simulations demonstrate that both finite-variance and infinite-variance noise benefit the fidelity of quantum teleportation.

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## 1. Introduction

Noise can sometimes benefit the detection of weak signals [1]. Researchers have dubbed this counterintuitive phenomenon as the *stochastic resonance* (SR) effect [2]. A typical performance curve displays that performance is poor for low noise values, increases to a maximal value for some optimal noise and tapers down again when too much noise is present (figure 2 displays such performance curves).

The SR noise benefit occurs in a diverse range of systems from neurons [3] to superconducting quantum interference devices [4] to crayfish [5]. The SR noise benefit also occurs

in the quantum regime with unique quantum effects such as squeezed light [6, 7], tunneling [8], quantum jumps in a micromaser [9], electron shelving [10] and entanglement [11]. All the aforementioned classical and quantum scenarios for SR involve the detection of weak signals.

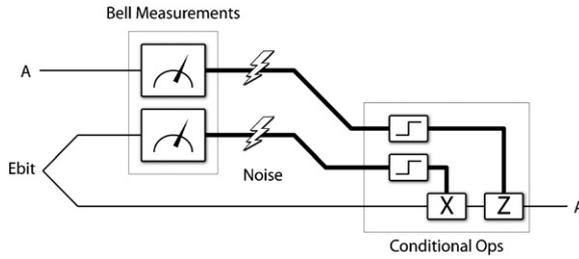
The ingredients for a noise benefit are weak signals, a nonlinear detection scheme and a source of noise energy. Noise energy does not benefit communication in linear systems because amplification only increases the noise in the signal. But small amounts of noise energy can be beneficial in a simple nonlinear threshold detection scheme. It can boost the signal above a threshold when it otherwise would be undetectable. The noise benefit occurs in most nonlinear systems because they act as threshold systems at some level.

The classical forbidden-interval theorems [12, 13] apply to a simple threshold system. The theorems give necessary and sufficient conditions for an SR noise benefit in a memoryless threshold neuron. The communication model for the theorems has a simple form in terms of a threshold function with threshold  $T$  and subthreshold bipolar signals with values  $-A$  and  $A$  where  $-A < 0 < A < T$ . The forbidden-interval condition is that an SR noise benefit occurs if and only if the noise mean does not lie in the interval  $(T - A, T + A)$ . The significance of the theorems is that the forbidden-interval condition implies that the SR noise benefit occurs in a memoryless threshold neuron for *any* finite-variance noise or infinite-variance alpha-stable noise [14].

I broaden the scope of the classical forbidden-interval theorems by constructing a modified teleportation protocol in which classical noise enhances the fidelity of quantum teleportation (figure 1). This phenomenon is an SR noise benefit because the enhancement occurs for some optimal non-zero classical noise level. The original quantum teleportation protocol uses one ebit of shared entanglement and two noiseless feedforward classical bits to transmit one qubit [15]. Later work considers a noisy teleportation protocol that sends quantum information with a noisy classical channel [16]. I consider a similar teleportation model where the entanglement is noiseless and the classical communication is over a noisy classical channel. But in this protocol, the transmitter Alice encodes the two classical bits as two weak, subthreshold, bipolar classical signals and sends them over the noisy channel. A receiver Bob then thresholds to determine the two classical bits Alice sent. This modified teleportation protocol then leads to an SR noise benefit for the fidelity of quantum communication.

The fidelity for the modified teleportation protocol qualitatively behaves similarly to the mutual information measure for the classical SR noise benefit in neurons in [12, 13]. The similarity is qualitative because the fidelity displays the full inverted- $U$  signature of the SR noise benefit given satisfaction of the forbidden-interval condition. But the fidelity measures quantum communication performance while the mutual information measures classical communication performance. Theorems 1 and 2 have the same forbidden-interval condition but now apply to the fidelity measure.

Two forbidden-interval theorems—theorems 1 and 2—give necessary and sufficient conditions for the SR noise benefit in quantum transmission. The first theorem holds for any finite-variance noise and the second theorem holds for infinite-variance alpha-stable noise. The proof strategy for theorems 1 and 2 is the same as the earlier strategy in [12, 13]. The original proof strategy in [12, 13] constructed all crucial arguments in terms of detection probabilities. The simple and elegant expression for the fidelity in lemma 2 in terms of detection probabilities implies that the same proof strategy is applicable. The proof strategy is to show that the fidelity must increase from its minimum of  $1/2$  with the addition of classical noise. This increase occurs if the fidelity approaches its minimum of  $1/2$  as the variance or the dispersion of the noise decreases to zero. In section 4, I show that the SR effect occurs even when the entanglement resource is imperfect, i.e., if the sender and receiver share



**Figure 1.** Modified quantum teleportation protocol with a noisy classical channel and thresholding. Thick lines denote classical information and thin lines denote quantum information. Alice wants to teleport quantum state  $A$  to Bob. Alice and Bob share an ebit. Alice receives two classical bits from a Bell measurement of her state  $A$  and her half of the shared ebit. Alice encodes the two classical bits as weak bipolar signals and transmits them over a noisy channel. Bob thresholds the signal he receives to retrieve two classical bits. Bob then performs a conditional rotation of his state by  $\hat{X}$  or  $\hat{Z}$  or both in the hope that he rotates his state to be Alice’s original state  $A$ .

noisy entanglement. Theorems 1 and 2 provide a theoretical underpinning to explain why the SR effect occurs in the modified teleportation protocol just as the original forbidden-interval theorems explain why the SR effect occurs in a noisy threshold neuron.

Ting has previously considered the SR effect in quantum communication [17–19]. He specifically considered the response of the coherent information and the fidelity to noise in several types of Pauli channels. He found that the coherent information quantum information measure does not exhibit a noise-enhanced SR effect, but the fidelity does exhibit such an effect. The present work is similar to his because I consider the fidelity of quantum communication as the measure of performance, but the model under which the stochastic resonance effect occurs is significantly different because I employ a modified teleportation protocol with subthreshold classical signals, while he considered the effect of transmitting qubits over noisy qubit channels.

## 2. A model for stochastic resonance in quantum teleportation

I first review the noiseless quantum teleportation protocol [15] before presenting the modified teleportation protocol. Suppose that Alice and Bob share one ebit—a maximally entangled quantum state  $|\Phi^+\rangle^{AB} \equiv (|0\rangle^A |0\rangle^B + |1\rangle^A |1\rangle^B)/\sqrt{2}$ . Suppose Alice wants to transmit a quantum bit  $|\psi\rangle^{A'} = \alpha |0\rangle^{A'} + \beta |1\rangle^{A'}$  to Bob. Define quantum state  $|\phi\rangle^{A'AB}$  as the joint state of system  $A'$  and ebit  $|\Phi^+\rangle^{AB}$ :

$$|\phi\rangle^{A'AB} \equiv |\psi\rangle^{A'} \otimes |\Phi^+\rangle^{AB}. \tag{1}$$

Alice can teleport state  $A'$  to Bob by performing a two-qubit Bell measurement on qubit  $A'$  and on her half  $A$  of the shared ebit. Alice receives two classical bits  $s_1 s_2$  from the Bell measurement where  $\forall i \in \{1, 2\}, s_i \in \{0, 1\}$ . The Bell measurement is probabilistic so that bits  $s_1$  and  $s_2$  are realizations of two Bernoulli random variables  $S_1$  and  $S_2$ , respectively. The following useful lemma gives the density of the two classical bits  $s_1 s_2$  that Alice receives from the Bell measurement. Lemma 1 follows simply from the original teleportation protocol [15]. The appendix gives the proof of the following lemma and of all following lemmas and theorems.

**Lemma 1.** *Random variables  $S_1$  and  $S_2$  from the Bell measurement are independent and identically distributed with equal probability of being zero or one:*

$$P_{S_i}(s_i) = 1/2 \forall i \in \{1, 2\} \forall s_i \in \{0, 1\}. \quad (2)$$

Alice transmits the two classical bits  $s_1, s_2$  over a noiseless classical channel. Bob receives the two classical bits and performs a conditional rotation  $\hat{Z}^{s_2} \hat{X}^{s_1}$  on his half  $B$  of the shared ebit.  $\hat{Z}$  and  $\hat{X}$  are the Pauli operators [20]. The teleportation is a perfect success if Alice can perform a perfect Bell measurement, if Alice sends two *noiseless* classical bits to Bob, and if Bob can perform the conditional unitaries without any small error in the rotation. The state in Bob's lab  $B$  is  $|\psi\rangle^B = \alpha |0\rangle^B + \beta |1\rangle^B$  when teleportation is perfect.

I now construct a modified teleportation protocol that uses *subthreshold* classical signals (figure 1). This model leads to an SR noise benefit for the fidelity of teleportation. Suppose Alice still performs a perfect two-qubit Bell measurement on her qubit  $|\psi\rangle^{A'}$  and her half  $A$  of the shared ebit. Alice receives two random classical bits  $s_1$  and  $s_2$  from the Bell measurement. Let  $S$  be a Bernoulli random variable with equal probability for outcome zero or outcome one. Bits  $s_1$  and  $s_2$  are independent realizations of random variable  $S$  by lemma 1. Suppose Alice cannot transmit noiseless classical bits and must instead use a continuous additive noisy classical channel for transmission [21]. Suppose further that Alice sends two weak, bipolar, subthreshold signals over the additive noisy classical channel. She encodes the random bits with the map  $(-1)^{S+1} A$  so that signal value  $-A$  corresponds to '0' and signal value  $A$  corresponds to '1'. The signals are weak in the sense that they are *subthreshold*—the threshold  $\theta$  is larger than the signal values:  $-A < 0 < A < \theta$ . The additive noisy channel corrupts the two classical signals by adding a random noise  $N$ . Suppose the noise  $N$  for two uses of the channel is independent and identically distributed. The noise  $N$  and random variable  $S$  are independent because the noise  $N$  plays no role in the Bell measurement. The two signals Bob receives from both uses of the channel are independent realizations of random variable  $(-1)^{S+1} A + N$ . Suppose Bob detects the classical signals by thresholding with a threshold  $\theta$ . He counts a '1' if the signal he receives is greater than  $\theta$  and counts a '0' if the signal is less than  $\theta$ . Let  $y_1$  and  $y_2$  be the two bits from Bob's detection. Both bits are independent realizations of a random variable  $Y$  where

$$Y = u((-1)^{S+1} A + N - \theta), \quad (3)$$

and  $u$  is the unit step or Heaviside function. The quantum state Bob possesses after Alice performs the Bell measurement is  $|\psi_{s_1, s_2}\rangle^B \equiv \hat{Z}^{s_2} \hat{X}^{s_1} |\psi\rangle^B$ . Bob does not have knowledge of bits  $s_1$  and  $s_2$  so he cannot rotate his state to be the same as Alice's original qubit  $|\psi\rangle^A$  with probability one. He can perform a rotation of his state based only on bits  $y_1$  and  $y_2$ . So Bob performs a conditional rotation  $\hat{Z}^{y_2} \hat{X}^{y_1}$  in an attempt to rotate the state  $|\psi_{s_1, s_2}\rangle^B$  to state  $|\psi\rangle^B$ . Suppose Bob performs a noiseless Pauli  $\hat{Z}$ ,  $\hat{X}$ , or  $\hat{Z}\hat{X}$  gate when he performs the conditional rotation. His resulting state is  $|\psi_{y_1, y_2, s_1, s_2}\rangle^B \equiv \hat{Z}^{y_2} \hat{X}^{y_1} |\psi_{s_1, s_2}\rangle^B$ . He does not apply the proper rotation if  $y_1 y_2 \neq s_1 s_2$ . Thus Bob's state is a mixture  $\rho_B$  equal to the following matrix:

$$\sum_{\substack{y_1, y_2, \\ s_1, s_2 = 0}}^1 p_{Y_1, Y_2, S_1, S_2}(y_1, y_2, s_1, s_2) |\psi_{y_1, y_2, s_1, s_2}\rangle \langle \psi_{y_1, y_2, s_1, s_2}|, \quad (4)$$

where  $p_{Y_1, Y_2, S_1, S_2}$  is the joint probability distribution of random variables  $Y_1, Y_2, S_1$  and  $S_2$  (we evaluate it later on). The modified teleportation protocol leads to noisy quantum communication because Bob's final state is the noisy mixed state above. Alice cannot teleport her state perfectly to Bob in the modified teleportation protocol with Alice encoding with subthreshold signals and Bob detecting with a threshold system.

The fidelity measure quantifies the quality of Alice and Bob’s quantum communication [20]. The fidelity  $F$  is as follows:

$$F \equiv \langle \psi | \rho_B | \psi \rangle, \tag{5}$$

where  $|\psi\rangle$  is Alice’s original state  $|\psi\rangle^{A'}$  and  $\rho_B$  is Bob’s mixed state from (4). Several example values of the fidelity elucidate some meaning behind this measure of quantum communication. The fidelity  $F = 1$  if and only if Alice’s state is the same as Bob’s state.  $F = 0$  if and only if Alice and Bob’s states are orthogonal. Suppose Bob ignores the classical information Alice sends in the teleportation protocol, randomly chooses a rotation and does not record which rotation he performs. Then his state is maximally mixed with density matrix  $\rho_B = I/2$ . So Alice and Bob’s fidelity  $F = 1/2$  if  $\rho_B = I/2$ . Alice and Bob can obtain a fidelity of teleportation equal to  $2/3$  even when they do not share entanglement and use only noiseless classical communication [22].

The fidelity for the modified teleportation protocol admits a simple mathematical form in terms of four quantities:  $q_X, q_Z, q_{XZ}$  and  $P$ . Define  $q_X, q_Z, q_{XZ}$  as

$$q_Z \equiv |\langle \psi | \hat{Z} | \psi \rangle|^2, \tag{6}$$

$$q_X \equiv |\langle \psi | \hat{X} | \psi \rangle|^2, \tag{7}$$

$$q_{XZ} \equiv |\langle \psi | \hat{X} \hat{Z} | \psi \rangle|^2. \tag{8}$$

The quantities  $q_X, q_Z$  and  $q_{XZ}$  depend on the probability amplitudes  $\alpha, \beta$  of Alice’s state  $|\psi\rangle^{A'}$ . The quantities  $q_X, q_Z$  and  $q_{XZ}$  are nonnegative and convex so that  $q_X + q_Z + q_{XZ} = 1$ . Define the nonnegative quantity  $P$  as the difference of conditional probabilities:

$$P \equiv p_{Y|S}(0|0) - p_{Y|S}(0|1) = p_{Y|S}(1|1) - p_{Y|S}(1|0). \tag{9}$$

The proof of the nonnegativity of  $P$  and the equality of the above conditional probability differences is in the proof of lemma 2. Note that equality of  $p_{Y|S}(0|0) - p_{Y|S}(0|1)$  and  $p_{Y|S}(1|1) - p_{Y|S}(1|0)$  holds because the classical signals in the model are subthreshold. As a simple example of this equality, note that  $p_{Y|S}(1|1) = 0, p_{Y|S}(1|0) = 0, p_{Y|S}(0|0) = 1$  and  $p_{Y|S}(0|1) = 1$  if there is no noise on the classical channel. Consider that lemma 2 gives the simple mathematical expression for the fidelity  $F$ .

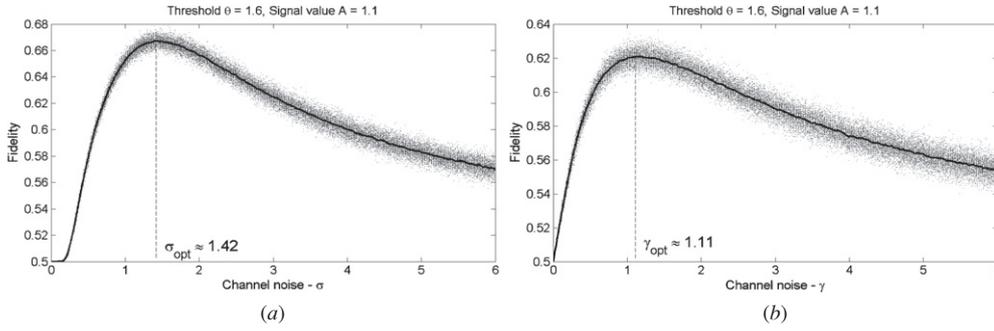
**Lemma 2.** *The fidelity  $F$  between Alice’s initial quantum state  $|\psi\rangle^{A'}$  and Bob’s mixed state  $\rho_B$  is*

$$F = \frac{1}{2} + \frac{P(q_X + q_Z + q_{XZ}P)}{2}, \tag{10}$$

*given the modified teleportation protocol.*

The noisy classical channel affects only parameter  $P$  and parameter  $P$  varies between zero and one depending on how much noise is present in the channel. The other parameters  $q_X, q_Z$  and  $q_{XZ}$  depend on the quantum state  $|\psi\rangle^{A'}$  that Alice wishes to teleport—they depend on the probability amplitudes  $\alpha, \beta$ . The noisy channel does not affect  $q_X, q_Z$  and  $q_{XZ}$  so that the fidelity changes with the noisiness of the channel regardless of the quantum state that Alice teleports.

The mutual information measure for classical SR has a more complicated relationship with parameter  $P$  than does the above fidelity measure [12, 13]. It is elegant that the fidelity measure for quantum communication has such a simple quadratic relation with parameter  $P$  given the modified teleportation protocol. Corollary 1 relates the fidelity of teleportation

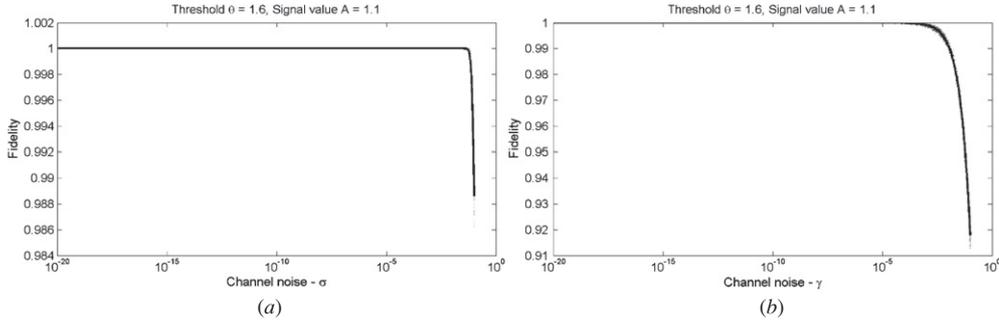


**Figure 2.** Stochastic resonance in the modified teleportation protocol. Alice possesses the state  $(|0\rangle + |1\rangle)/\sqrt{2}$  and wishes to teleport it to Bob. The graphs show the smoothed teleportation fidelity (thick line) and min–max deviation (dotted lines) as a function of (a) the variance of classical Gaussian noise and a function of (b) the dispersion of classical Cauchy noise for 100 simulation runs. Alice encodes bipolar signals with amplitude  $A = 1.1$  and Bob decodes with threshold  $\theta = 1.6$ . Each run generated 10 000 input–output signal pairs to estimate the fidelity of teleportation. Graph (a) is a simulation instance of the if-part of theorem 1 with finite-variance Gaussian noise. The noise mean  $\mu = 0$  and lies outside the forbidden interval  $(0.5, 2.7)$ . The average teleportation fidelity exceeds the classical limit of  $2/3$  [22] with  $F = 0.6682$  for a noise standard deviation  $\sigma_{\text{opt}} = 1.42$ . Graph (b) is a simulation instance of the if-part of theorem 2 with infinite-variance Cauchy noise. The noise location  $a = 0$  and lies outside the forbidden interval  $(0.5, 2.7)$ . The average teleportation fidelity does not exceed the classical limit of  $2/3$  with  $F = 0.6213$  for a noise dispersion  $\gamma_{\text{opt}} = 1.11$ .

to the statistical relationship between random variables  $S$  and  $Y$ . The relationship follows by determining the quantity  $P$  when random variables  $S$  and  $Y$  are statistically dependent, statistically independent and when  $S$  and  $Y$  correlate perfectly. The relationship of the fidelity  $F$  with random variables  $S$  and  $Y$  follows directly from the relationship of parameter  $P$  with  $S$  and  $Y$  by using (10).

**Corollary 1.** *The fidelity  $F$  between Alice’s initial quantum state  $|\psi\rangle^{A'}$  and Bob’s mixed state  $\rho_B$  is minimum at  $1/2$  given the modified teleportation protocol. The fidelity  $F$  obtains this minimum value if and only if random variable  $Y$  is independent of random variable  $S$ . The fidelity  $F > 1/2$  if  $Y$  and  $S$  are statistically dependent. The fidelity  $F$  is equal to its maximum of one when detection is perfect.*

Corollary 1 is useful because it provides both a lower and upper bound for the fidelity of teleportation given the modified teleportation protocol. It also gives the scenarios in which these lower and upper bounds saturate. The fidelity cannot decrease below  $1/2$  for any amount of noise in the classical channel. This lower bound is a powerful way to characterize the stochastic resonance effect in terms of the fidelity. The SR noise benefit has a nonmonotone signature because the performance measure decreases as the noise level decreases. So the fidelity should decrease to its minimum of  $1/2$  when the noise variance or dispersion of the channel decreases to zero. This statement is equivalent to saying that the fidelity increases from its minimum value of  $1/2$  as the noise variance or dispersion of the channel increases: *what goes down must come up*. The if-part of the theorems employ the *what goes down must come up* strategy to show that the fidelity approaches its minimum of  $1/2$  when the noise vanishes similar to the way that the mutual information approaches its minimum of zero when the noise vanishes [12]. Corollary 1 is also useful because it gives the situation in which the fidelity is equal to its maximum of one. This situation provides a powerful way



**Figure 3.** No stochastic resonance when the noise mean or location lies in the forbidden interval. Alice possesses the state  $(|0\rangle + |1\rangle)/\sqrt{2}$  and wishes to teleport it to Bob. The graphs show the smoothed teleportation fidelity (thick line) and min–max deviation (dotted lines) as a function of (a) the variance of classical Gaussian noise and a function of (b) the dispersion of classical Cauchy noise for 100 simulation runs. Alice encodes bipolar signals with amplitude  $A = 1.1$  and Bob decodes with threshold  $\theta = 1.6$ . Each run generated 10 000 input–output signal pairs to estimate the fidelity of teleportation. Graph (a) is a simulation instance of the only-if part of theorem 1 with finite-variance Gaussian noise. The noise mean  $\mu = 0.7$  and lies inside the forbidden interval  $(0.5, 2.7)$  so that no SR occurs. Graph (b) is a simulation instance of the only-if part of theorem 2 with infinite-variance Cauchy noise. The noise location  $a = 0.7$  and lies inside the forbidden interval  $(0.5, 2.7)$  so that no SR occurs.

of determining when the SR noise benefit does not occur. The SR noise benefit does not occur when the fidelity of teleportation increases to its maximum value of one as the noise variance or dispersion decreases to zero. The only-if part of the theorems show that the fidelity approaches its maximum value of one as the noise vanishes similar to the way that the mutual information approaches its maximum of one as the noise vanishes [13]. I employ these proof strategies involving the lower and upper bounds from corollary 1 in the proofs of the main results: theorems 1 and 2.

### 3. Forbidden-interval theorems for quantum teleportation

#### 3.1. SR with finite-variance noise

Theorem 1 below characterizes the nonmonotone SR noise benefit when the classical channel noise has finite variance. Theorem 1 states that the modified teleportation protocol exhibits the SR effect if and only if the classical noise mean obeys an interval constraint. The noise mean must lie outside a forbidden interval that depends on Bob’s detection threshold  $\theta$  and signal value  $A$ . The teleportation fidelity defined in (10) quantifies the SR noise benefit. Figure 2(a) is a simulation instance of the if-part of theorem 1 and figure 3(a) is a simulation instance of the only-if part of theorem 1 when the classical channel noise is Gaussian distributed. The significance of theorem 1 is that it holds for any finite-variance noise regardless of the particular density of the noise.

**Theorem 1.** *Suppose that the channel noise has finite variance  $\sigma^2$  and mean  $\mu$ . Suppose that there is some statistical dependence between Alice’s classical signal  $S$  and Bob’s threshold result  $Y$  so that the fidelity obeys  $F > 1/2$ . Then the quantum teleportation system features the nonmonotone SR effect if and only if the noise mean does not lie in the forbidden interval:  $\mu \notin (\theta - A, \theta + A)$ . The nonmonotone SR effect is that  $F \rightarrow 1/2$  as  $\sigma^2 \rightarrow 0$ .*

### 3.2. SR with infinite-variance noise

The uncountably infinite family of alpha-stable noise densities models many diverse physical phenomena that include impulsive interrupts in phone lines, underwater acoustics, low-frequency atmospheric signals and gravitational fluctuations [14]. The parameter  $\alpha$  for the alpha-stable noise density lies in the interval  $(0, 2]$ . It characterizes the thickness of the curve's tails:  $\alpha = 1$  corresponds to the thick-tailed Cauchy random variable and  $\alpha = 2$  corresponds to the familiar thin-tailed Gaussian random variable. The curve's tail thickness increases as  $\alpha$  decreases. The generalized central limit theorem states that all and only normalized stable random variables converge in distribution to a stable random variable [23]. The characteristic function  $\varphi(\omega)$  of a general alpha-stable random variable is

$$\varphi(\omega) = \exp \left\{ i a \omega - \gamma |\omega|^\alpha \left( 1 + i \beta \text{sign}(\omega) \tan \left( \frac{\alpha \pi}{2} \right) \right) \right\}, \quad (11)$$

for  $\alpha \neq 1$  and

$$\varphi(\omega) = \exp \{ i a \omega - \gamma |\omega| (1 - 2i \beta \text{sign}(\omega) \ln(|\omega|/\pi)) \}, \quad (12)$$

for  $\alpha = 1$  where

$$\text{sign}(\omega) = \begin{cases} 1 : \omega > 0 \\ 0 : \omega = 0 \\ -1 : \omega < 0 \end{cases}, \quad (13)$$

and  $i = \sqrt{-1}$ ,  $0 < \alpha \leq 2$ ,  $-1 \leq \beta \leq 1$ , and  $\gamma > 0$ . Parameter  $\beta$  is a skewness parameter where  $\beta = 0$  gives a symmetric density. Theorem 2 holds for any skewness  $\beta$ . Parameter  $\gamma$  is a dispersion parameter similar in spirit to the variance. It quantifies the spread or width of the alpha-stable density. Thick-tailed alpha-stable noise may corrupt Alice's classical bipolar signals if she sends them over an impulsive phone line or as a low-frequency signal through the atmosphere.

Theorem 2 characterizes the nonmonotone SR noise benefit when the classical channel noise has an infinite-variance alpha-stable density. Figure 2(b) is a simulation instance of the if-part of theorem 2 and figure 3(b) is a simulation instance of the only-if part of theorem 2 when the classical channel noise is infinite-variance Cauchy distributed. Theorem 2 demonstrates that the SR noise benefit for quantum communication is robust because it occurs even in situations when the classical noise has infinite variance.

**Theorem 2.** *Suppose the channel noise has an infinite-variance alpha-stable density with dispersion  $\gamma$  and location  $a$ . Suppose that there is some statistical dependence between Alice's classical signal  $S$  and Bob's threshold result  $Y$  so that the fidelity obeys  $F > 1/2$ . Then the quantum teleportation system features the nonmonotone SR effect if and only if the noise location does not lie in the forbidden interval:  $a \notin (\theta - A, \theta + A)$ . The nonmonotone SR effect is that  $F \rightarrow 1/2$  as  $\gamma \rightarrow 0$ .*

## 4. Imperfect entanglement

The entanglement shared between Alice and Bob may not always be perfect, and it is natural to wonder whether the SR effect still occurs. I briefly show that variations of the above forbidden-interval theorems hold for the more realistic case where the shared entanglement is in an imperfectly entangled Werner-like state [24]. Thus, the SR effect still occurs when the entanglement is imperfect.

Let us now suppose that Alice and Bob share the following Werner-like state as the entanglement resource for teleportation:

$$\rho_W = F_W |\Phi^+\rangle\langle\Phi^+|^{AB} + (1 - F_W) \pi^A \otimes \pi^B,$$

where  $\pi$  is the maximally mixed qubit state. We can interpret the above state as being a perfectly entangled ebit with probability  $F_W$  and being in a completely mixed state with probability  $1 - F_W$ .

Let us consider using the above imperfectly entangled resource for the modified teleportation protocol. Suppose that Alice and Bob perform the modified teleportation protocol. It is straightforward to show that Bob's resulting state is as it was before with probability  $F_W$  and it is the completely mixed state with probability  $1 - F_W$ . Then, omitting the details, the resulting expression for the fidelity is

$$\begin{aligned} F &= F_W \left( \frac{1}{2} + \frac{P(q_X + q_Z + q_{XZ}P)}{2} \right) + \frac{1 - F_W}{2} \\ &= \frac{1}{2} + \frac{F_W P(q_X + q_Z + q_{XZ}P)}{2}. \end{aligned}$$

The above expression is similar to that we obtained before, with the difference that the fidelity now depends on the parameter  $F_W$  from the Werner-like state. Thus, the fidelity of teleportation in this 'imperfect entanglement' scenario still bears the SR signature because we can apply all of the above forbidden-interval theorems.

## 5. Conclusion

The theorems for the SR noise benefit prove that small amounts of noise can enhance the fidelity of quantum teleportation given the modified teleportation protocol. The theorems lend credence to the conjecture in [12] that an SR noise benefit should occur in any nonlinear system whose input–output structure is a threshold system. The theorems show that the SR effect is robust because it occurs for all finite-variance noise types and for infinite-variance alpha-stable noise.

The theorems do not guarantee a specific performance for the teleportation fidelity. They do not even guarantee that the teleportation fidelity exceeds the classical limit. Figure 2(b) is an example of a failure to exceed the classical limit of 2/3 due to impulsive Cauchy noise. The theorems guarantee only that performance with noise exceeds performance without noise given the satisfaction of the forbidden-interval condition.

Some may question whether the modified teleportation protocol leads to a true 'quantum' stochastic resonance. It is after all not quantum noise that affects the fidelity in this model but rather classical noise. But several quantum effects are present in the modified teleportation protocol such as entanglement, Bell measurements and the coherence of the quantum state being teleported. The interplay of quantum effects with the noisy classical channel argues that we should categorize this result as a *classical-noise-assisted quantum stochastic resonance*.

The theorems also suggest that the SR noise benefit will occur in any quantum protocol that uses feedforward classical communication with subthreshold signals. Protocols such as entanglement purification, distillation, gate teleportation [25] and the Knill–Laflamme–Milburn scheme for linear optical quantum computation [26] all require classical signals. Any quantum protocol with feedforward memoryless classical communication should exhibit the SR noise benefit when the sender transmits subthreshold classical signals over a noisy channel and the receiver performs threshold detection.

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## Appendix. Proofs

**Proof (Lemma 1).** Define states  $|\Phi^+\rangle^{A'A}$ ,  $|\Phi^-\rangle^{A'A}$ ,  $|\Psi^+\rangle^{A'A}$  and  $|\Psi^-\rangle^{A'A}$  as the Bell basis states:

$$|\Phi^+\rangle^{A'A} \equiv \frac{|0\rangle^{A'}|0\rangle^A + |1\rangle^{A'}|1\rangle^A}{\sqrt{2}}, \quad (\text{A.1})$$

$$|\Phi^-\rangle^{A'A} \equiv \frac{|0\rangle^{A'}|0\rangle^A - |1\rangle^{A'}|1\rangle^A}{\sqrt{2}}, \quad (\text{A.2})$$

$$|\Psi^+\rangle^{A'A} \equiv \frac{|0\rangle^{A'}|1\rangle^A + |1\rangle^{A'}|0\rangle^A}{\sqrt{2}}, \quad (\text{A.3})$$

$$|\Psi^-\rangle^{A'A} \equiv \frac{|0\rangle^{A'}|1\rangle^A - |1\rangle^{A'}|0\rangle^A}{\sqrt{2}}. \quad (\text{A.4})$$

Make the following additional assignments:  $|\Phi^+\rangle^{A'A} \equiv |\Phi_{00}\rangle^{A'A}$ ,  $|\Phi^-\rangle^{A'A} \equiv |\Phi_{01}\rangle^{A'A}$ ,  $|\Psi^+\rangle^{A'A} \equiv |\Phi_{10}\rangle^{A'A}$ ,  $|\Psi^-\rangle^{A'A} \equiv |\Phi_{11}\rangle^{A'A}$ . Define the rotated states  $|\psi_{00}\rangle^B$ ,  $|\psi_{01}\rangle^B$ ,  $|\psi_{10}\rangle^B$  and  $|\psi_{11}\rangle^B$  as follows:

$$|\psi_{00}\rangle^B \equiv |\psi\rangle^B \equiv \alpha |0\rangle^B + \beta |1\rangle^B, \quad (\text{A.5})$$

$$|\psi_{01}\rangle^B \equiv \alpha |0\rangle^B - \beta |1\rangle^B = \hat{Z} |\psi\rangle^B, \quad (\text{A.6})$$

$$|\psi_{10}\rangle^B \equiv \alpha |1\rangle^B + \beta |0\rangle^B = \hat{X} |\psi\rangle^B, \quad (\text{A.7})$$

$$|\psi_{11}\rangle^B \equiv \alpha |1\rangle^B - \beta |0\rangle^B = \hat{X}\hat{Z} |\psi\rangle^B. \quad (\text{A.8})$$

Write the joint state  $|\phi\rangle^{A'AB}$  from (1) in the following form by performing a few algebraic steps [15]:

$$|\phi\rangle^{A'AB} = \frac{1}{2} \begin{bmatrix} |\Phi_{00}\rangle^{A'A} \otimes |\psi_{00}\rangle^B + \\ |\Phi_{01}\rangle^{A'A} \otimes |\psi_{01}\rangle^B + \\ |\Phi_{10}\rangle^{A'A} \otimes |\psi_{10}\rangle^B + \\ |\Phi_{11}\rangle^{A'A} \otimes |\psi_{11}\rangle^B \end{bmatrix}. \quad (\text{A.9})$$

Alice performs a measurement in the Bell basis on her two qubits  $A'$  and  $A$ . The joint state  $|\phi\rangle^{A'AB}$  collapses to one of four states in the set  $\{|\Phi_{00}\rangle^{A'A} \otimes |\psi_{00}\rangle^B, |\Phi_{01}\rangle^{A'A} \otimes |\psi_{01}\rangle^B, |\Phi_{10}\rangle^{A'A} \otimes |\psi_{10}\rangle^B, |\Phi_{11}\rangle^{A'A} \otimes |\psi_{11}\rangle^B\}$ . Alice's measurement in the Bell basis gives two classical bits  $s_1 s_2$  given by the subscripts in the above equation. Suppose the two bits  $s_1$  and  $s_2$  are realizations of two Bernoulli random variables  $S_1$  and  $S_2$ , respectively. Squaring the probability amplitudes of each state in the superposition of quantum state  $|\phi\rangle^{A'AB}$  gives

an equal probability of 1/4 for each possible state resulting from the Bell measurement. The joint distribution of  $S_1$  and  $S_2$  is as follows:

$$p_{S_1, S_2}(s_1, s_2) = \frac{1}{4} \quad \forall s_1, s_2 \in \{0, 1\}. \quad (\text{A.10})$$

Thus both  $S_1$  and  $S_2$  are uniform random variables with the same density. The marginal probabilities must also be uniform:

$$p_{S_1}(s_1) = \frac{1}{2} \quad \forall s_1 \in \{0, 1\}, \quad (\text{A.11})$$

$$p_{S_2}(s_2) = \frac{1}{2} \quad \forall s_2 \in \{0, 1\}. \quad (\text{A.12})$$

$S_1$  and  $S_2$  are independent random variables because the joint density is the product of the marginals. □

**Proof (Lemma 2).** Random variables  $Y_1$  and  $S_1$ , and  $Y_2$  and  $S_2$  are independent because of the reasoning in section 2. Consider the joint density  $p_{Y_1, Y_2, S_1, S_2}(y_1, y_2, s_1, s_2)$  for Bob's two random variables  $Y_1$  and  $Y_2$  and Alice's two random variables  $S_1$  and  $S_2$ :

$$p_{Y_1, Y_2, S_1, S_2}(y_1, y_2, s_1, s_2) \quad (\text{A.13})$$

$$= p_{Y_1, S_1 | Y_2, S_2}(y_1, s_1 | y_2, s_2) p_{Y_2, S_2}(y_2, s_2) \quad (\text{A.14})$$

$$= p_{Y_1, S_1}(y_1, s_1) p_{Y_2, S_2}(y_2, s_2) \quad (\text{A.15})$$

$$= p_{Y_1 | S_1}(y_1 | s_1) p_{S_1}(s_1) p_{Y_2 | S_2}(y_2 | s_2) p_{S_2}(s_2) \quad (\text{A.16})$$

$$= p_{Y_1 | S_1}(y_1 | s_1) \left(\frac{1}{2}\right) p_{Y_2 | S_2}(y_2 | s_2) \left(\frac{1}{2}\right) \quad (\text{A.17})$$

$$= \frac{1}{4} p_{Y | S}(y_1 | s_1) p_{Y | S}(y_2 | s_2). \quad (\text{A.18})$$

Consider the projectors  $|\psi_{y_1, y_2, s_1, s_2}\rangle\langle\psi_{y_1, y_2, s_1, s_2}|$  in Bob's mixed state  $\rho_B$  from (4):

$$|\psi_{y_1, y_2, s_1, s_2}\rangle\langle\psi_{y_1, y_2, s_1, s_2}| \quad (\text{A.19})$$

$$= \hat{Z}^{y_2} \hat{X}^{y_1} |\psi_{s_1, s_2}\rangle\langle\psi_{s_1, s_2}| \hat{X}^{y_1} \hat{Z}^{y_2} \quad (\text{A.20})$$

$$= \hat{Z}^{y_2} \hat{X}^{y_1} \hat{X}^{s_1} \hat{Z}^{s_2} |\psi\rangle\langle\psi| \hat{Z}^{s_2} \hat{X}^{s_1} \hat{X}^{y_1} \hat{Z}^{y_2} \quad (\text{A.21})$$

$$= \hat{Z}^{y_2} \hat{X}^{y_1 \oplus s_1} \hat{Z}^{s_2} |\psi\rangle\langle\psi| \hat{Z}^{s_2} \hat{X}^{y_1 \oplus s_1} \hat{Z}^{y_2} \quad (\text{A.22})$$

$$= \left( \begin{array}{c} (-1)^{y_1 \oplus s_1} \hat{X}^{y_1 \oplus s_1} \hat{Z}^{y_2} \hat{Z}^{s_2} \\ |\psi\rangle\langle\psi| \hat{Z}^{s_2} \hat{Z}^{y_2} (-1)^{y_1 \oplus s_1} \hat{X}^{y_1 \oplus s_1} \end{array} \right) \quad (\text{A.23})$$

$$= \hat{X}^{y_1 \oplus s_1} \hat{Z}^{y_2 \oplus s_2} |\psi\rangle\langle\psi| \hat{Z}^{y_2 \oplus s_2} \hat{X}^{y_1 \oplus s_1} \quad (\text{A.24})$$

$$= |\psi_{y_1 \oplus s_1, y_2 \oplus s_2}\rangle\langle\psi_{y_1 \oplus s_1, y_2 \oplus s_2}|. \quad (\text{A.25})$$

So Bob's mixed state  $\rho_B$  is as follows by substituting (A.18) for the joint density and substituting (A.25) for the projectors  $|\psi_{y_1, y_2, s_1, s_2}\rangle\langle\psi_{y_1, y_2, s_1, s_2}|$ :

$$\rho_B = \frac{1}{4} \sum_{\substack{y_1, y_2, \\ s_1, s_2 = 0}}^1 \left( \begin{array}{c} p_{Y | S}(y_1 | s_1) p_{Y | S}(y_2 | s_2) \\ |\psi_{y_1 \oplus s_1, y_2 \oplus s_2}\rangle\langle\psi_{y_1 \oplus s_1, y_2 \oplus s_2}| \end{array} \right). \quad (\text{A.26})$$

Now use the above expression for Bob's mixed state to compute the fidelity  $F$  between Alice's original state  $|\psi\rangle^{A'}$  and Bob's mixed state  $\rho_B$ :

$$F = \langle \psi | \rho_B | \psi \rangle \quad (\text{A.27})$$

$$= \left\langle \psi \left| \frac{1}{4} \sum_{\substack{y_1, y_2, \\ s_1, s_2 = 0}}^1 \left( p_{Y|S}(y_1|s_1) p_{Y|S}(y_2|s_2) \right) \left| \psi_{y_1 \oplus s_1, y_2 \oplus s_2} \right\rangle \right| \psi \right\rangle \quad (\text{A.28})$$

$$= \frac{1}{4} \sum_{\substack{y_1, y_2, \\ s_1, s_2 = 0}}^1 \left( p_{Y|S}(y_1|s_1) p_{Y|S}(y_2|s_2) \right) \left| \langle \psi | \psi_{y_1 \oplus s_1, y_2 \oplus s_2} \rangle \right|^2. \quad (\text{A.29})$$

The quantity  $|\langle \psi | \psi_{y_1 \oplus s_1, y_2 \oplus s_2} \rangle|^2$  can take one of the following four values depending on the bit values  $y_1, y_2, s_1$  and  $s_2$ :

$$|\langle \psi | \hat{Z} | \psi \rangle|^2 = |\alpha|^4 - 2|\alpha|^2|\beta|^2 + |\beta|^4, \quad (\text{A.30})$$

$$|\langle \psi | \hat{X} | \psi \rangle|^2 = 2 \left( |\beta|^2 |\alpha|^2 + \text{Re} \left\{ \beta^2 (\alpha^*)^2 \right\} \right), \quad (\text{A.31})$$

$$|\langle \psi | \hat{X} \hat{Z} | \psi \rangle|^2 = 2(|\beta|^2 |\alpha|^2 - \text{Re} \{ \beta^2 (\alpha^*)^2 \}), \quad (\text{A.32})$$

$$|\langle \psi | \psi \rangle|^2 = 1 = |\alpha|^4 + 2|\alpha|^2|\beta|^2 + |\beta|^4. \quad (\text{A.33})$$

Define the nonnegative quantities  $q_Z, q_X$  and  $q_{XZ}$  as in (6)–(8). The quantities  $q_Z, q_X$  and  $q_{XZ}$  sum to one using (A.30)–(A.33):  $q_Z + q_X + q_{XZ} = 1$ . Use the following shorthand for the conditional probabilities:

$$p_{y_1|s_1} \equiv p_{Y|S}(y_1|s_1), \quad (\text{A.34})$$

$$p_{y_2|s_2} \equiv p_{Y|S}(y_2|s_2). \quad (\text{A.35})$$

I now prove that conditional probability differences  $p_{0|0} - p_{0|1}$  and  $p_{1|1} - p_{1|0}$  are nonnegative and equal to each other. Consider the conditional probability  $p_{0|0}$ :

$$p_{0|0} = p_{Y|S}(0|0) \quad (\text{A.36})$$

$$= \Pr \{ u((-1)^{S+1} A + N - \theta) = 0 | S = 0 \} \quad (\text{A.37})$$

$$= \Pr \{ u(-A + N - \theta) = 0 \} \quad (\text{A.38})$$

$$= \Pr \{ -A + N - \theta < 0 \} \quad (\text{A.39})$$

$$= \Pr \{ N < \theta + A \} \quad (\text{A.40})$$

$$= \int_{-\infty}^{\theta+A} p_N(n) \, dn, \quad (\text{A.41})$$

where  $p_N(n)$  is the density of the noise  $N$ . The other three conditional probabilities follow from similar reasoning:

$$p_{0|1} = \int_{-\infty}^{\theta-A} p_N(n) \, dn, \quad (\text{A.42})$$

$$p_{1|1} = \int_{\theta-A}^{\infty} p_N(n) \, dn, \quad (\text{A.43})$$

$$p_{1|0} = \int_{\theta+A}^{\infty} p_N(n) dn. \tag{A.44}$$

So the conditional probability differences are equal and nonnegative because  $p_N(n)$  is nonnegative:

$$p_{0|0} - p_{0|1} = \int_{\theta-A}^{\theta+A} p_N(n) dn, \tag{A.45}$$

$$p_{1|1} - p_{1|0} = \int_{\theta-A}^{\theta+A} p_N(n) dn. \tag{A.46}$$

Define the nonnegative quantity  $P \equiv p_{0|0} - p_{0|1} = p_{1|1} - p_{1|0}$ . Let us return to the proof of the fidelity expression. Expand the fidelity  $F$  from (A.29) as follows:

$$= \frac{1}{4} \begin{pmatrix} p_{0|0}p_{0|0} + p_{0|0}p_{0|1}q_Z + p_{0|1}p_{0|0}q_X + \\ p_{0|1}p_{0|1}q_{XZ} + p_{0|0}p_{1|0}q_Z + p_{0|0}p_{1|1} + \\ p_{0|1}p_{1|0}q_{XZ} + p_{0|1}p_{1|1}q_X + p_{1|0}p_{0|0}q_X + \\ p_{1|0}p_{0|1}q_{XZ} + p_{1|1}p_{0|0} + p_{1|1}p_{0|1}q_Z + \\ p_{1|0}p_{1|0}q_{XZ} + p_{1|0}p_{1|1}q_X + p_{1|1}p_{1|0}q_Z + \\ p_{1|1}p_{1|1} \end{pmatrix} \tag{A.47}$$

$$= \frac{1}{4} \begin{pmatrix} p_{0|0}p_{0|0} + p_{0|0}p_{1|1} + \\ p_{1|1}p_{0|0} + p_{1|1}p_{1|1} + \\ \left[ \begin{matrix} p_{0|0}p_{1|0} + p_{0|0}p_{0|1} + \\ p_{1|1}p_{0|1} + p_{1|1}p_{1|0} \end{matrix} \right] q_Z + \\ \left[ \begin{matrix} p_{0|1}p_{0|0} + p_{0|1}p_{1|1} + \\ p_{1|0}p_{0|0} + p_{1|0}p_{1|1} \end{matrix} \right] q_X + \\ \left[ \begin{matrix} p_{0|1}p_{0|1} + p_{0|1}p_{1|0} + \\ p_{1|0}p_{0|1} + p_{1|0}p_{1|0} \end{matrix} \right] q_{XZ} \end{pmatrix} \tag{A.48}$$

$$= \frac{1}{4} \begin{pmatrix} p_{0|0}p_{0|0} + p_{0|0}p_{1|1} + \\ p_{1|1}p_{0|0} + p_{1|1}p_{1|1} + \\ \left[ \begin{matrix} p_{0|0}p_{1|0} + p_{0|0}p_{0|1} + \\ p_{1|1}p_{0|1} + p_{1|1}p_{1|0} \end{matrix} \right] (q_Z + q_X) + \\ \left[ \begin{matrix} p_{0|1}p_{0|1} + p_{0|1}p_{1|0} + \\ p_{1|0}p_{0|1} + p_{1|0}p_{1|0} \end{matrix} \right] q_{XZ} \end{pmatrix} \tag{A.49}$$

$$= \frac{1}{4} \begin{pmatrix} p_{0|0}p_{0|0} + p_{0|0}p_{1|1} + \\ p_{1|1}p_{0|0} + p_{1|1}p_{1|1} + \\ \left[ \begin{matrix} p_{0|0}p_{1|0} + p_{0|0}p_{0|1} + \\ p_{1|1}p_{0|1} + p_{1|1}p_{1|0} \end{matrix} \right] (1 - q_{XZ}) + \\ \left[ \begin{matrix} p_{0|1}p_{0|1} + p_{0|1}p_{1|0} + \\ p_{1|0}p_{0|1} + p_{1|0}p_{1|0} \end{matrix} \right] q_{XZ} \end{pmatrix} \tag{A.50}$$

$$= \frac{1}{4} \begin{pmatrix} p_{0|0} + p_{0|0} + \\ p_{1|1} + p_{1|1} + \\ \left[ \begin{matrix} Pp_{0|1} + Pp_{1|0} + \\ Pp_{0|1} + Pp_{1|0} \end{matrix} \right] (-q_{XZ}) \end{pmatrix} \tag{A.51}$$

$$= \frac{1}{4} \left( \begin{array}{c} 2 + 2P \\ + [2P (p_{0|1} + p_{1|0})] (-q_{XZ}) \end{array} \right) \quad (\text{A.52})$$

$$= (2 + 2P (1 - q_{XZ} (p_{0|1} + p_{1|0}))) / 4 \quad (\text{A.53})$$

$$= (1 + P (1 - q_{XZ} (1 - p_{1|1} + p_{1|0}))) / 2 \quad (\text{A.54})$$

$$= (1 + P (1 - q_{XZ} + q_{XZ}P)) / 2 \quad (\text{A.55})$$

$$= \frac{1}{2} + \frac{P (q_X + q_Z + q_{XZ}P)}{2}. \quad (\text{A.56})$$

The quantities  $q_X + q_Z$ ,  $q_{XZ}$ ,  $P$  in the above expression are nonnegative.  $\square$

**Proof (Corollary 1).** First characterize the relationship between random variables  $S$  and  $Y$  and parameter  $P$ . Then translate this relationship to the fidelity  $F$  by using (10). Expand the probability  $p_Y(y)$  using the law of total probability:

$$p_Y(y) = p_{Y|S}(y|0) p_S(0) + p_{Y|S}(y|1) p_S(1) \quad (\text{A.57})$$

$$= p_{Y|S}(y|0) p_S(0) + p_{Y|S}(y|1) (1 - p_S(0)) \quad (\text{A.58})$$

$$= (p_{Y|S}(y|0) - p_{Y|S}(y|1)) p_S(0) + p_{Y|S}(y|1). \quad (\text{A.59})$$

Consider when  $y = 0$ :

$$p_Y(0) = (p_{Y|S}(0|0) - p_{Y|S}(0|1)) p_S(0) + p_{Y|S}(0|1) \quad (\text{A.60})$$

$$= P p_S(0) + p_{Y|S}(0|1). \quad (\text{A.61})$$

The probability  $p_Y(0) = p_{Y|S}(0|1)$  when parameter  $P = 0$ . Expand the probability  $p_Y(y)$  in a similar manner so that

$$p_Y(y) = (p_{Y|S}(y|1) - p_{Y|S}(y|0)) p_S(1) + p_{Y|S}(y|0). \quad (\text{A.62})$$

Consider when  $y = 1$ :

$$p_Y(1) = (p_{Y|S}(1|1) - p_{Y|S}(1|0)) p_S(1) + p_{Y|S}(1|0) \quad (\text{A.63})$$

$$P p_S(1) + p_{Y|S}(1|0). \quad (\text{A.64})$$

The probability  $p_Y(1) = p_{Y|S}(1|0)$  when parameter  $P = 0$ . Random variables  $Y$  and  $S$  are independent if and only if parameter  $P = 0$  because the probabilities  $p_Y(0)$  and  $p_Y(1)$  are equal to probabilities conditioned on  $S$ . Detection is perfect if and only if the conditional probabilities  $p_{Y|S}(0|0) = p_{Y|S}(1|1) = 1$ . Suppose  $P = 1$ . Then

$$1 = p_{Y|S}(0|0) - p_{Y|S}(0|1) \quad (\text{A.65})$$

$$= p_{Y|S}(0|0) - (1 - p_{Y|S}(1|1)) \quad (\text{A.66})$$

$$= p_{Y|S}(0|0) - 1 + p_{Y|S}(1|1) \quad (\text{A.67})$$

$$\Leftrightarrow 2 = p_{Y|S}(0|0) + p_{Y|S}(1|1). \quad (\text{A.68})$$

Both  $p_{Y|S}(0|0) = p_{Y|S}(1|1) = 1$  because they are probabilities and neither  $p_{Y|S}(0|0)$  nor  $p_{Y|S}(1|1)$  can be greater than one. So detection is perfect if and only if  $P = 1$ . Parameter  $P$  varies between zero and one. The fidelity  $F$  becomes minimum at  $1/2$  if  $P$  vanishes by using

(10). A nonzero value of  $P$  corresponds to statistical dependence of random variables  $S$  and  $Y$  and gives a fidelity  $F > 1/2$ . Perfect detection gives  $P = 1$  and gives a perfect fidelity  $F = 1$  because  $q_X, q_Z$  and  $q_{XZ}$  are nonnegative and sum to one using (A.30)–(A.33).  $\square$

**Proof (Theorem 1).** Suppose the noise mean  $\mu$  is not in the forbidden interval:  $\mu \notin (\theta - A, \theta + A)$ . Then I prove that  $P$  vanishes when the finite variance  $\sigma^2 \rightarrow 0$ . The fidelity  $F$  approaches its minimum of  $1/2$  when  $P \rightarrow 0$ . Thus the fidelity  $F$  rises from its minimum at  $1/2$  as the channel adds some noise. The ‘what goes down must come up’ proof strategy is the same as the earlier forbidden-interval theorem proofs [12, 13]. I include the full proof for completeness. I first prove the sufficient condition. Ignore the zero-measure case when  $\mu = \theta + A$  or  $\mu = \theta - A$ . Suppose the noise mean  $\mu > \theta + A$ . Pick  $\varepsilon = (\mu - \theta - A) / 2 > 0$  so that  $\theta + A + \varepsilon = \mu - \varepsilon$ . Consider parameter  $P$ :

$$P = \int_{\theta-A}^{\theta+A} p_N(n) \, dn \tag{A.69}$$

$$\leq \int_{-\infty}^{\theta+A} p_N(n) \, dn \tag{A.70}$$

$$\leq \int_{-\infty}^{\theta+A+\varepsilon} p_N(n) \, dn \tag{A.71}$$

$$= \int_{-\infty}^{\mu-\varepsilon} p_N(n) \, dn \tag{A.72}$$

$$= \Pr\{N < \mu - \varepsilon\} \tag{A.73}$$

$$= \Pr\{N - \mu < -\varepsilon\} \tag{A.74}$$

$$\leq \Pr\{|N - \mu| > \varepsilon\} \tag{A.75}$$

$$\leq \frac{\sigma^2}{\varepsilon^2} \rightarrow 0 \text{ as } \sigma^2 \rightarrow 0. \tag{A.76}$$

Suppose the noise mean  $\mu < \theta - A$ . Pick  $\varepsilon = (\theta - A - \mu) / 2 > 0$  so that  $\theta - A - \varepsilon = \mu + \varepsilon$ . Consider parameter  $P$ :

$$P = \int_{\theta-A}^{\theta+A} p_N(n) \, dn \tag{A.77}$$

$$\leq \int_{\theta-A}^{\infty} p_N(n) \, dn \tag{A.78}$$

$$\leq \int_{\theta-A-\varepsilon}^{\infty} p_N(n) \, dn \tag{A.79}$$

$$= \int_{\mu+\varepsilon}^{\infty} p_N(n) \, dn \tag{A.80}$$

$$= \Pr\{N > \mu + \varepsilon\} \tag{A.81}$$

$$= \Pr\{N - \mu > \varepsilon\} \tag{A.82}$$

$$\leq \Pr\{|N - \mu| > \varepsilon\} \tag{A.83}$$

$$\leq \frac{\sigma^2}{\varepsilon^2} \rightarrow 0 \text{ as } \sigma^2 \rightarrow 0. \tag{A.84}$$

I now prove that the forbidden-interval condition is necessary for the SR noise benefit. Suppose the noise mean  $\mu$  is in the forbidden interval:  $\mu \in (\theta - A, \theta + A)$ . Then I prove that parameter  $P \rightarrow 1$  as  $\sigma^2 \rightarrow 0$  and thus the fidelity  $F \rightarrow 1$  by corollary 1. So the nonmonotone SR noise benefit does not occur as the noise variance vanishes. Parameter  $P \rightarrow 1$  if and only if the conditional probabilities  $p_{Y|S}(0|0) \rightarrow 1$  and  $p_{Y|S}(1|1) \rightarrow 1$ . Consider the conditional probability  $p_{Y|S}(0|0)$ . Pick  $\varepsilon = (\theta + A - \mu)/2$ . Then  $\theta + A - \varepsilon = \mu + \varepsilon$ .

$$p_{Y|S}(0|0) = \int_{-\infty}^{\theta+A} p_N(n) \, dn \tag{A.85}$$

$$\geq \int_{-\infty}^{\theta+A-\varepsilon} p_N(n) \, dn \tag{A.86}$$

$$= \int_{-\infty}^{\mu+\varepsilon} p_N(n) \, dn \tag{A.87}$$

$$= \Pr\{N < \mu + \varepsilon\} \tag{A.88}$$

$$= \Pr\{N - \mu < \varepsilon\} \tag{A.89}$$

$$= 1 - \Pr\{N - \mu \geq \varepsilon\} \tag{A.90}$$

$$\geq 1 - \Pr\{|N - \mu| \geq \varepsilon\} \tag{A.91}$$

$$\geq 1 - \frac{\sigma^2}{\varepsilon^2} \rightarrow 1 \text{ as } \sigma^2 \rightarrow 0. \tag{A.92}$$

Consider the conditional probability  $p_{Y|S}(1|1)$ . Pick  $\varepsilon = (\mu - \theta + A)/2$ . Then  $\theta - A + \varepsilon = \mu - \varepsilon$  so that

$$p_{Y|S}(1|1) = \int_{\theta-A}^{\infty} p_N(n) \, dn \tag{A.93}$$

$$\geq \int_{\theta-A+\varepsilon}^{\infty} p_N(n) \, dn \tag{A.94}$$

$$= \int_{\mu-\varepsilon}^{\infty} p_N(n) \, dn \tag{A.95}$$

$$= \Pr\{N > \mu - \varepsilon\} \tag{A.96}$$

$$= \Pr\{N - \mu > -\varepsilon\} \tag{A.97}$$

$$= 1 - \Pr\{N - \mu \leq -\varepsilon\} \tag{A.98}$$

$$\geq 1 - \Pr\{|N - \mu| \geq \varepsilon\} \tag{A.99}$$

$$\geq 1 - \frac{\sigma^2}{\varepsilon^2} \rightarrow 1 \text{ as } \sigma^2 \rightarrow 0. \tag{A.100}$$

So parameter  $P \rightarrow 1$  because the conditional probabilities  $p_{Y|S}(0|0) \rightarrow 1$  and  $p_{Y|S}(1|1) \rightarrow 1$ . The fidelity  $F \rightarrow 1$  as the noise vanishes and the nonmonotone SR noise benefit does not occur. □

**Proof (Theorem 2).** Suppose the noise location  $a$  is not in the forbidden interval:  $a \notin (\theta - A, \theta + A)$ . Then I prove that  $P$  vanishes when the dispersion  $\gamma \rightarrow 0$ . The fidelity  $F$  approaches its minimum of  $1/2$  when  $P \rightarrow 0$ . Thus the fidelity  $F$  rises from its minimum at  $1/2$  as the classical channel adds some noise. The alpha-stable proof strategy is the same as the earlier alpha-stable forbidden-interval theorem proofs [12, 13]. I include the full proof for completeness. The proof for the alpha-stable case is simple because the characteristic function in (11) and (12) approaches the following as the dispersion vanishes:

$$\lim_{\gamma \rightarrow 0} \varphi(\omega) = \exp(ia\omega). \tag{A.101}$$

The inverse Fourier transform of the characteristic function gives the limiting density as a translated delta function:  $\lim_{\gamma \rightarrow 0} p_N(n) = \delta(n - a)$ . I first prove the sufficient condition. Ignore the zero-measure case when  $\mu = \theta + A$  or  $\mu = \theta - A$ . Consider parameter  $P$ :

$$\begin{aligned} \lim_{\gamma \rightarrow 0} P &= \lim_{\gamma \rightarrow 0} \int_{\theta-A}^{\theta+A} p_N(n) \, dn \\ &= \int_{\theta-A}^{\theta+A} \delta(n - a) \, dn = 0. \end{aligned}$$

So the fidelity  $F$  approaches its minimum at  $1/2$  as the channel noise dispersion  $\gamma$  vanishes. I now prove that the forbidden-interval condition is necessary for the nonmonotone SR noise benefit. Suppose the noise location  $a$  is in the forbidden interval:  $a \in (\theta - A, \theta + A)$ . Then I prove that parameter  $P \rightarrow 1$  as  $\gamma \rightarrow 0$  and thus the fidelity  $F \rightarrow 1$  by corollary 1. So the nonmonotone SR effect does not occur as the noise dispersion vanishes. Parameter  $P \rightarrow 1$  if and only if the conditional probabilities  $p_{Y|S}(0|0) \rightarrow 1$  and  $p_{Y|S}(1|1) \rightarrow 1$ . Consider the conditional probability  $p_{Y|S}(0|0)$ :

$$\lim_{\gamma \rightarrow 0} p_{Y|S}(0|0) = \lim_{\gamma \rightarrow 0} \int_{-\infty}^{\theta+A} p_N(n) \, dn = \int_{-\infty}^{\theta+A} \delta(n - a) \, dn = 1.$$

Consider the conditional probability  $p_{Y|S}(1|1)$ :

$$\lim_{\gamma \rightarrow 0} p_{Y|S}(1|1) = \lim_{\gamma \rightarrow 0} \int_{\theta-A}^{\infty} p_N(n) \, dn = \int_{\theta-A}^{\infty} \delta(n - a) \, dn = 1.$$

So parameter  $P \rightarrow 1$  because the conditional probabilities  $p_{Y|S}(0|0) \rightarrow 1$  and  $p_{Y|S}(1|1) \rightarrow 1$ . The fidelity  $F \rightarrow 1$  as the noise vanishes and the nonmonotone SR noise benefit does not occur. □

## References

- [1] Bulsara A R and Gammaitoni L 1996 *Phys. Today* **49** 39
- [2] Gammaitoni L, Hanggi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223
- [3] Bulsara A R, Elston T C, Doering C R, Lowen S B and Lindenberg K 1996 *Phys. Rev. E* **53** 3958
- [4] Hibbs A, Jacobs E W, Bekkedahl J, Bulsara A R and Moss F 1995 *Il Nuovo Cimento* **17 D** 811
- [5] Wiesenfeld K and Moss F 1995 *Nature* **373** 33
- [6] Wilde M M and Kosko B 2006 *Proc. 8th Int. Conf. on Quantum Communication, Measurement, and Computing*
- [7] Wilde M M and Kosko B 2008 arXiv:0801.3141

- [8] Löfstedt R and Coppersmith S N 1994 *Phys. Rev. Lett.* **72** 1947
- [9] Buchleitner A and Mantegna R N 1998 *Phys. Rev. Lett.* **80** 3932
- [10] Huelga S F and Plenio M B 2000 *Phys. Rev. A* **62** 052111
- [11] Plenio M B and Huelga S F 2002 *Phys. Rev. Lett.* **88** 197901
- [12] Kosko B and Mitaim S 2003 *Neural Netw.* **14** 755
- [13] Kosko B and Mitaim S 2004 *Phys. Rev. E* **70** 031911
- [14] Nikias C and Shao M 1995 *Signal Processing with Alpha-Stable Distributions and Applications* (New York: Wiley)
- [15] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
- [16] Devetak I, Harrow A W and Winter A 2008 *IEEE Trans. Inf. Theory* **54** 4587
- [17] Ting J J-L 1999 *Phys. Lett. A* **259** 349
- [18] Ting J J-L 2000 *Eur. Phys. J. B* **13** 527
- [19] Ting J J-L 1999 *Phys. Rev. E* **59** 2801
- [20] Nielsen M and Chuang I 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [21] Cover T M and Thomas J A 1991 *Elements of Information Theory* (New York: Wiley)
- [22] Popescu S 1994 *Phys. Rev. Lett.* **72** 797
- [23] Breiman L 1968 *Probability* (Reading, MA: Addison-Wesley)
- [24] Werner R F 1989 *Phys. Rev. A* **40** 4277
- [25] Gottesman D and Chuang I 1999 *Nature* **402** 390
- [26] Knill E, Laflamme R and Milburn G 2001 *Nature* **409** 46