Quantum rate distortion, reverse Shannon theorems, and source-channel separation

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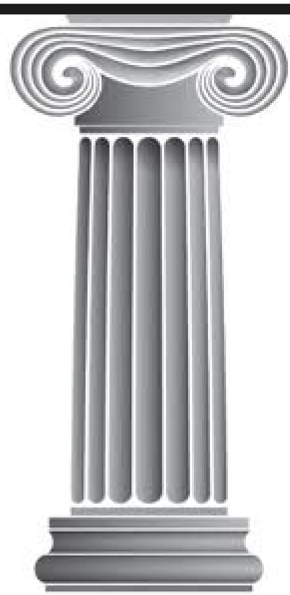
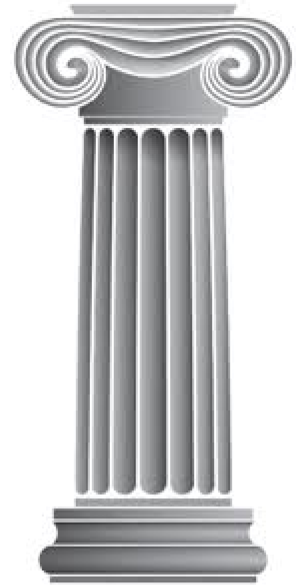
Classical Information Theory





Shannon's Source CodingTheorem

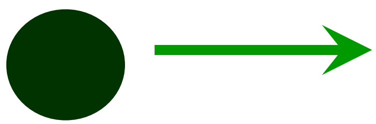
Compression of information emitted by a source

Shannon's Noisy Channel Coding Theorem

Transmission of information through a noisy channel

Shannon's Coding Theorems

Source Coding Theorem



memoryless

stochastically independent signals

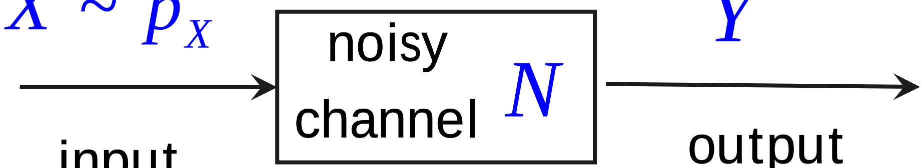
$$X \sim p_X \qquad ; \; x \in \mathcal{X}$$

The fundamental limit of data compression:

= *Shannon entropy of the source:*

$$H(X)$$

Channel Coding Theorem



The fundamental limit on the rate of reliable information transmission:

= *Capacity of the channel*

$$C(N) = \max_{p_X} I(X:Y)$$

mutual information

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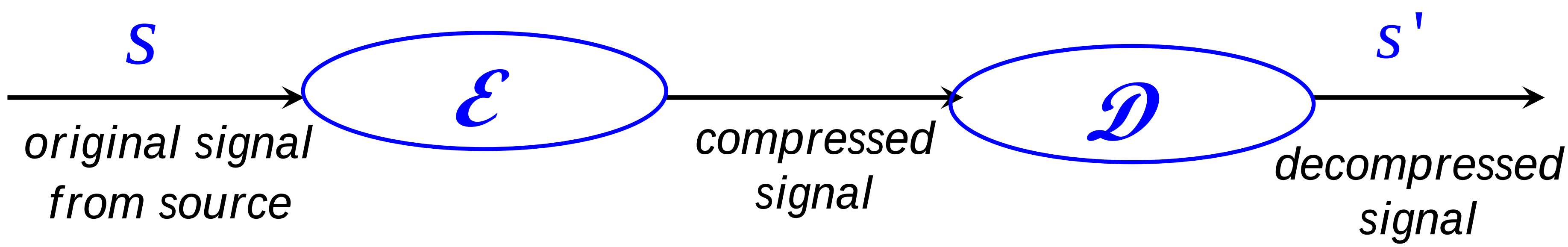
Lossless Data Compression



$s' = s$ (exact replica!)

Shannon's Source CodingTheorem

corresponds to asymptotically lossless data compression

- multiple (n) uses of the channel
- signals: sequences $x_1, x_2, \ldots, x_n$

$$\overline{p}_e(n) \rightarrow 0 \quad \text{as} \quad n \rightarrow 0$$

- (average probability of error in recovering original signal)

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Lossless Compression - often too stringent a condition

- for cases of multimedia data;

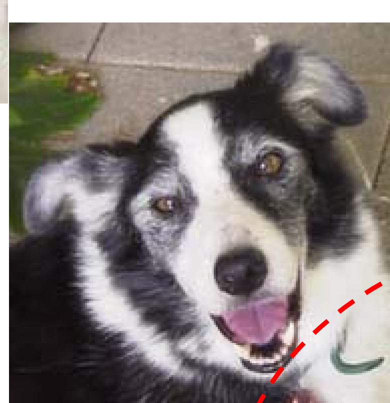
  e.g. audio, video and still images
- when storage space is insufficient

- **Why ?**

Typically, a substantial amount of data can be discarded before the information is sufficiently degraded to be noticeable!

An Example

original image (uncompressed), size 108.5 KB



lossless compression (PNG); size 60.1 KB

lossy compression; 4.82 KB

lossy compression; 1.14 KB

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Lossy Data Compression

- a data compression scheme :

- recovered data $\neq$ original data

instead

- recovered data $\overset{D}{\approx}$ original data

$D:$ the allowed distortion

Rate Distortion Theory $=$ the theory of Lossy Data Compression

- rate of compression $\overset{\text{tradeoff}}{\longleftrightarrow}$ the allowed distortion

→ fundamental limit on the asymptotic rate of data compression for a given maximum distortion

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Classical Rate Distortion Theory (Shannon)

- For a given memoryless information source ; $X \sim p_X$
- If the maximum allowed distortion is D ; $0 \leq D < 1$,

The minimum rate of data compression:

$$R(D) = \min_{(a)} I(X:Y) \leq H(X)$$

rate distortion function

mutual information

$$(a) : p_{Y|X} \quad ; \mathbf{E}(d(X,Y)) \leq D$$

stochastic maps

average distortion

Y : output of a stochastic map $p_{Y|X} : X \to Y$

$d(x, y)$: distortion measure e.g. $d(x,y) = (x-y)^2$

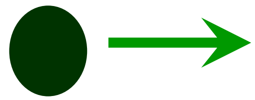
Our Aim

To obtain rate distortion functions in the quantum realm

Scenario for Quantum Rate Distortion

for a fixed $0 \leq D < 1$

- Storage setting



$\{\rho, \mathcal{H}_A\}$ a memoryless quantum information source

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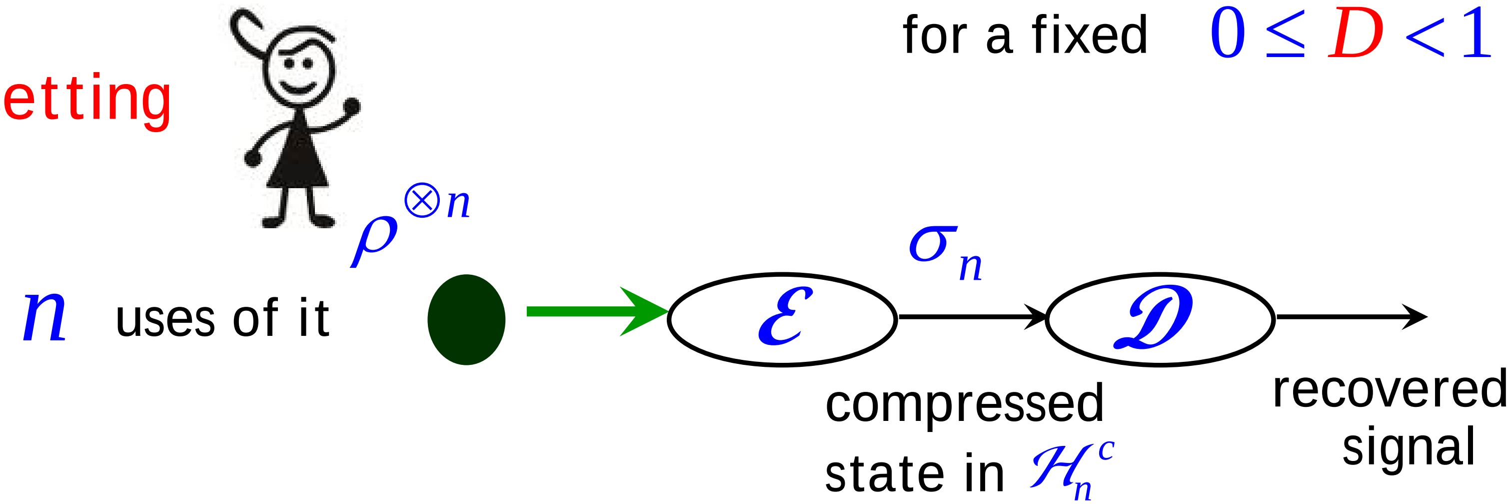
Scenario for Quantum Rate Distortion

for a fixed $0 \leq D < 1$

- Storage setting



Rate of data compression $= R$ *if* $\dim \mathcal{H}_n^c = 2^{nR}$

achievable – *if* $\lim_{n\to\infty} \bar{d}(\rho^{\otimes n}, \mathcal{D} \circ \mathcal{E}) \leq D$

$$R_q(D) := \inf\{R : \text{achievable}\}$$

Quantum Rate Distortion function

(min. rate of compression under given distortion)

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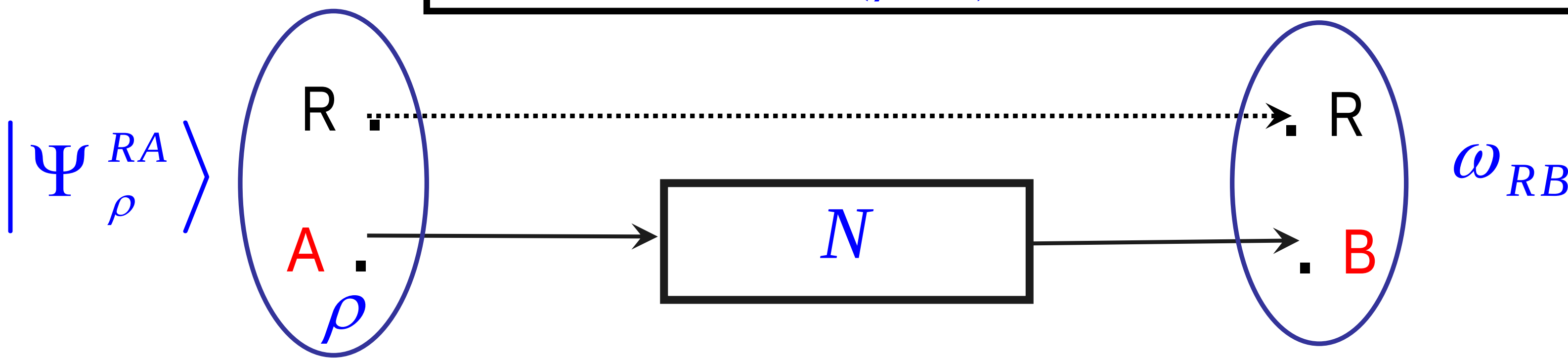
Quantum Rate Distortion Function

Barnum proved:

$$R_q(D) \geq \min_{\substack{N:\,\mathrm{CPTP} \\ d(\rho,N)\leq D}} I_c(\rho, N) \quad \ldots\ldots(1)$$

coherent information



$$I_c(\rho, N) = -S(R|B)_\omega; \qquad d(\rho, N) = 1 - F_e(\rho, N)$$

entanglement fidelity

- He conjectured: " $=$ " holds in (1)

problem!!

- the coherent information can be negative

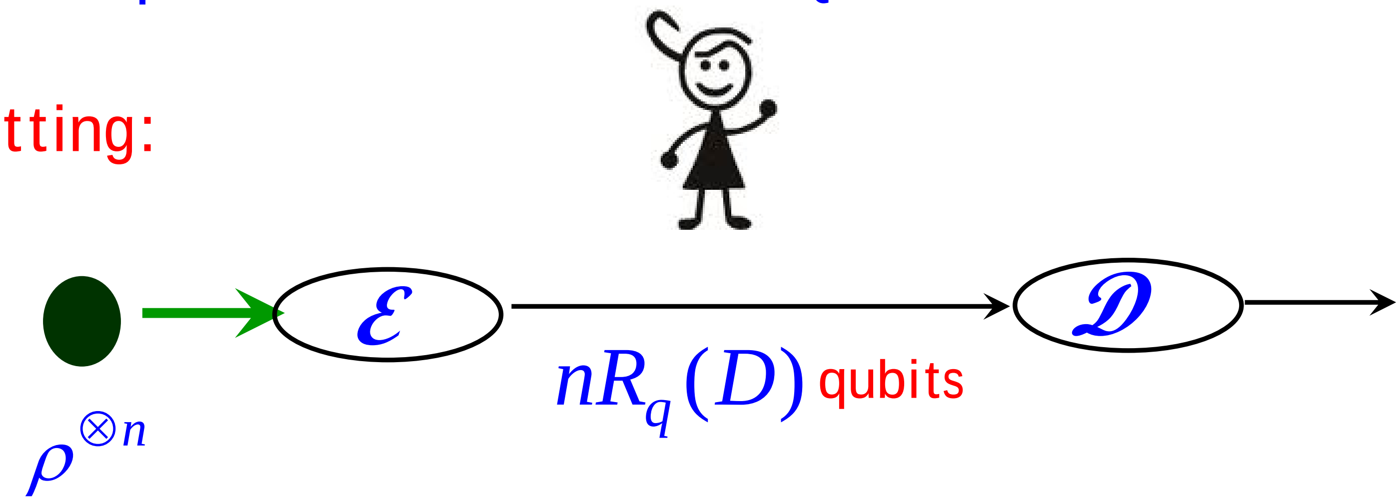
BUT

- The rate of a data compression scheme is non-negative !!

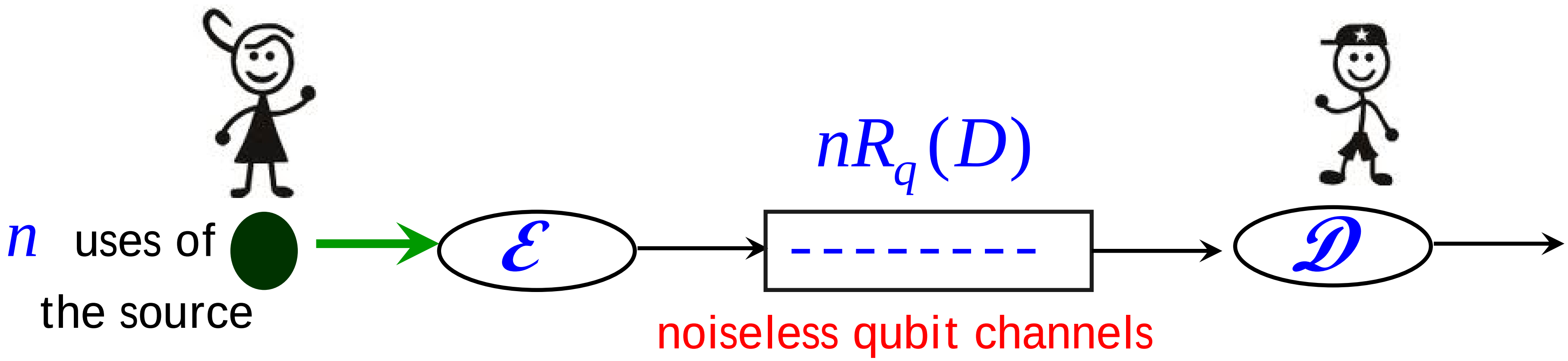
Equivalent Scenarios for Quantum Rate Distortion

- Storage setting:

n uses of the source

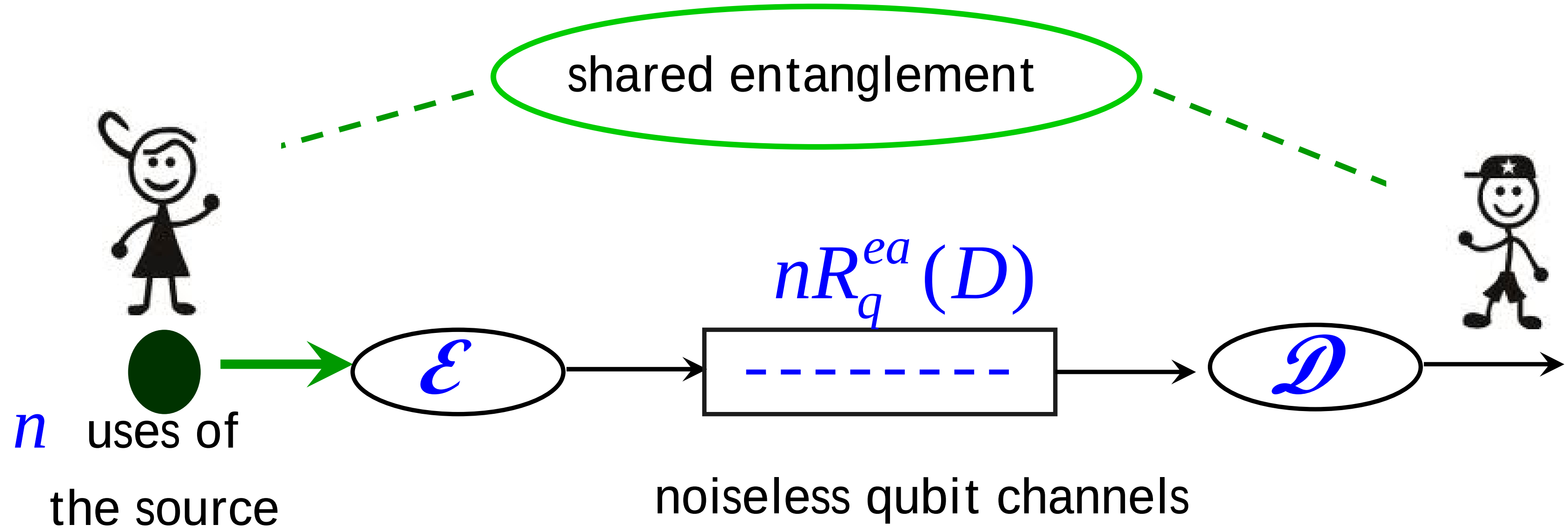


- Communication setting:



Another Scenario for Quantum Rate Distortion

- Communication setting: (entanglement assisted)



$R_q^{ea}(D) =$ *min. number of noiseless qubit channels needed per use of the source in the presence of entanglement for a given distortion* D

entanglement-assisted quantum rate distortion function

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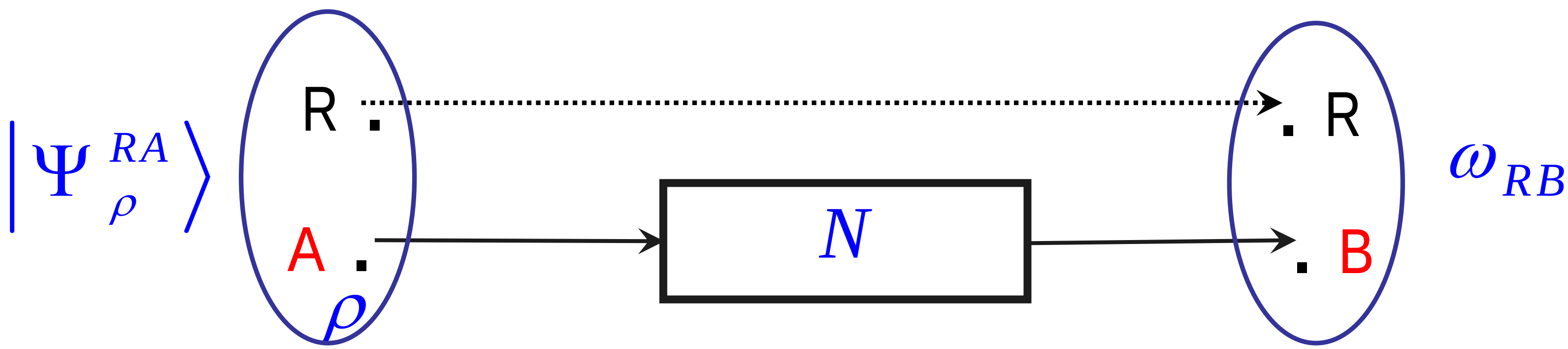
Result - I

- Entanglement-assisted quantum rate distortion function:

$$R_q^{ea}(D) = \min_{\substack{N:\,\text{CPTP} \\ d(\rho,N)\le D}} \frac{1}{2} I(R:B)_\omega \qquad \text{..........(2)}$$

quantum mutual information



- **Proof:**

(i) Prove : $R_q^{ea}(D) \geq$ RHS of (2) *using entropic inequalities*

(ii) Prove: This bound is achievable, i.e., " $=$ " holds

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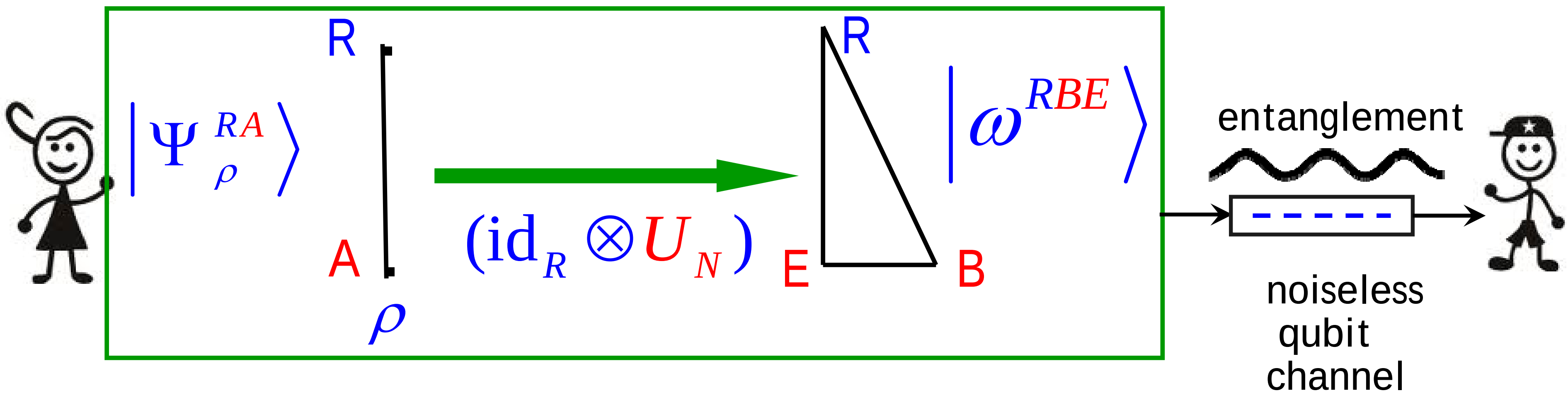
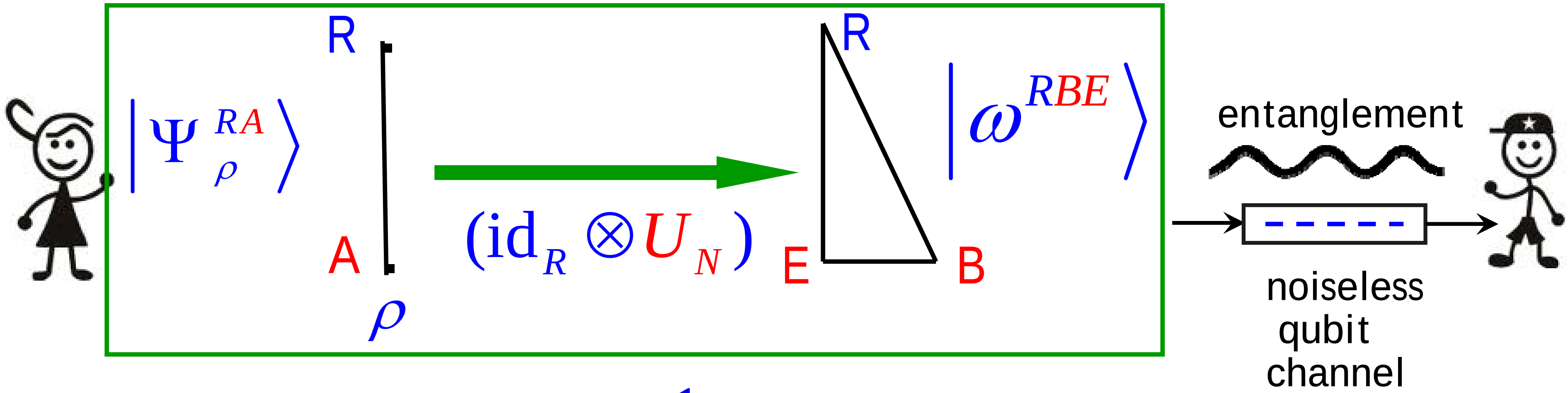
- The rate $R = \min\limits_{\substack{N:\,\mathrm{CPTP} \\ d(\rho,N)\leq D}} \frac{1}{2} I(R:B)_\omega$ is achievable

Sketch of proof:

- Source $\{\rho, \mathcal{H}_A\}$; Its purification: $|\Psi_\rho^{RA}\rangle$
- Let $N =$ the minimizing CPTP map
- Let $U_N : A \to BE$ a Stinespring isometry of N





To prove: $R = \min_{\substack{N:\,\mathrm{CPTP} \\ d(\rho,N)\le D}} \frac{1}{2} I(R:B)_\omega$ is achievable

- Note: $\omega^B := N(\rho)$ satisfies $d(\rho, N) \le D$

Suffices to prove:

ω_B can be sent to Bob by Alice through $\frac{1}{2} I(R:B)_\omega$ uses of a noiseless qubit channel per use of the source

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- How can we transfer ω_B to Bob ?

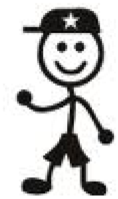
Asymptotic setting

- initially

$$\rho^{\otimes n} \xrightarrow[U_N^{\otimes n}]{} \omega_{BE}^{\otimes n}$$

State splitting

- finally



$$\sigma_n \approx \omega_B^{\otimes n}$$

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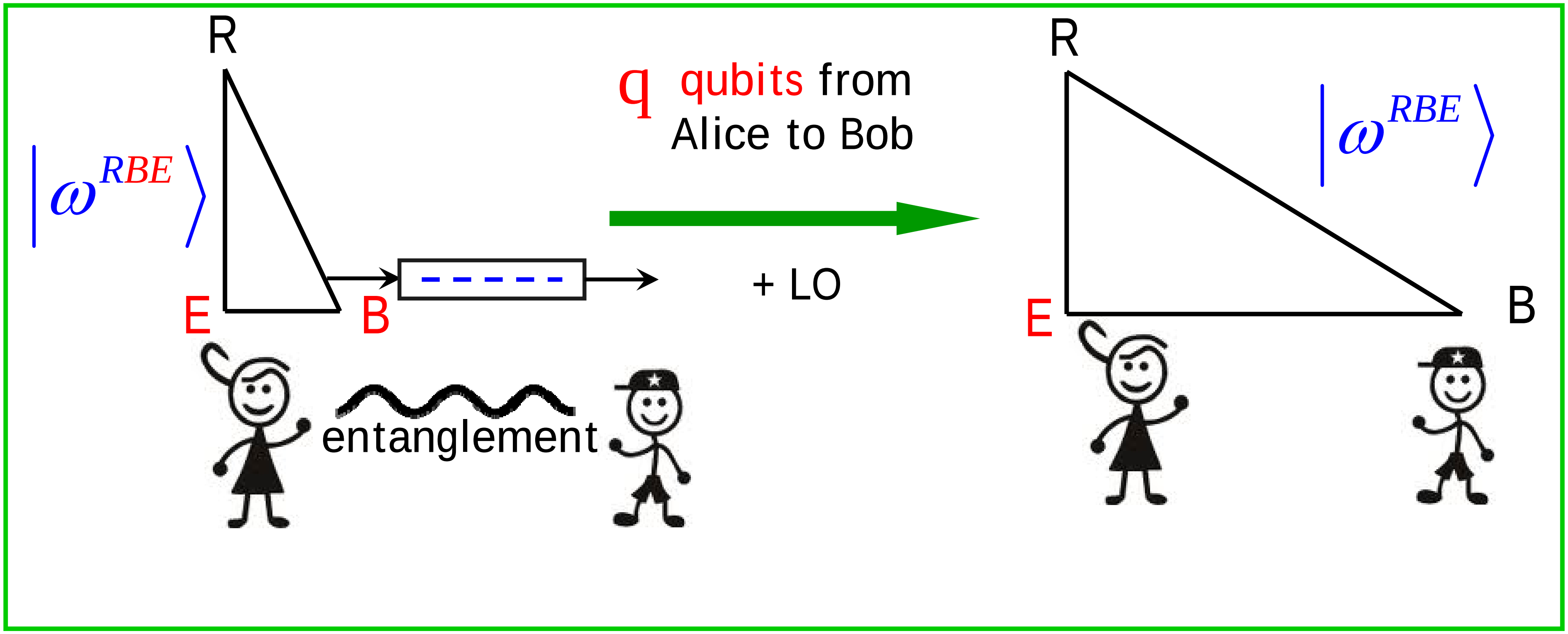
- How can we transfer ω_B to Bob ?



$$q = \frac{1}{2} I(R:B)_\omega = \min_{\substack{N:\,\mathrm{CPTP} \\ d(\rho,N)\le D}} \frac{1}{2} I(R:B)_\omega$$

achievable rate !

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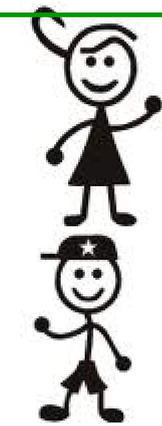
(i) local preparation of $\rho^{\otimes n} \xrightarrow[U_N^{\otimes n}]{} \omega_{BE}^{\otimes n}$

(ii) State splitting with the help of shared entanglement such that $\omega_B^{\otimes n}$



$\equiv$ simulating the (output of the) quantum channel $N^{\otimes n}$ when the input is $\rho^{\otimes n}$ (using shared entanglement)

$=$ a special case of another protocol -- channel simulation – Quantum Reverse Shannon Theorem **(QRST)**

$\Longrightarrow$ achievability of $R_q^{ea}(D)$

Result - II

- Unassisted Quantum rate distortion function:

$$R_q(D) = \lim_{n\to\infty}\frac{1}{n} \min_{\substack{N^{(n)}:\ \mathrm{CPTP} \\ d(\rho^{\otimes n}, N^{(n)})\le D}} E_P^{\infty}\left(N^{(n)}(\rho^{\otimes n})\right)$$

regularized entanglement of purification

- For any $\sigma_{AB} \to |\sigma_{ABR}\rangle \to \sigma_{BR}$

$$E_P(\sigma_{BR}) := \min_{\Lambda_R:CPTP} H\left((\mathrm{id}_B \otimes \Lambda_R)\sigma_{BR}\right) \ge 0$$

- where,

  for any state ρ, $H(\rho)$: Von Neumann entropy of ρ

Result - II

- Unassisted Quantum rate distortion function:

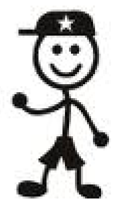
$$R_q(D) = \lim_{n\to\infty} \frac{1}{n} \min_{\substack{N:\ \text{CPTP} \\ d(\rho, N^{(n)}) \le D}} E_P^{\infty}\left(N^{(n)}(\rho^{\otimes n})\right)$$

Channel simulation (QRST) in the **absence** of shared entanglement $\Longrightarrow$ achievability of this rate

To simulate output of $N^{\otimes n}$ when input is $\rho^{\otimes n}$

qubits sent at a rate

$E_P^{\infty}(\omega_{RB})$



$\approx \omega_B^{\otimes n}$

$\omega_B = N(\rho)$

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Source Channel Separation Theorems

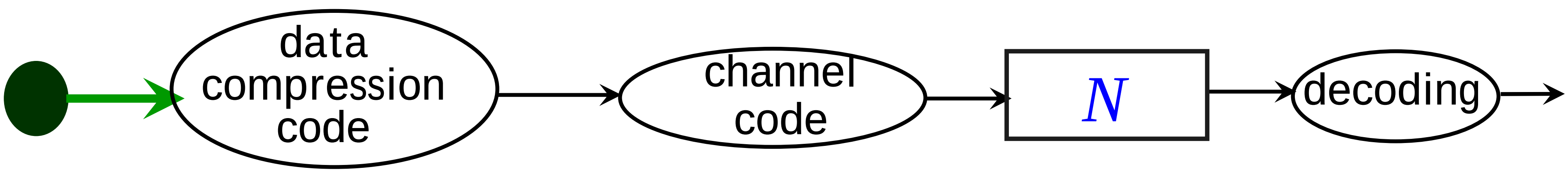
- **Shannon:** connects the two pillars of Classical Info. Theory

- Is it possible to transmit a classical information source (X) reliably over a classical channel (N) ?

- Yes -- if & only if

$$H(X) \leq C(N)$$

- **Implication:**



- a 2-stage encoding & decoding with the best data compression + error correction codes is optimal!

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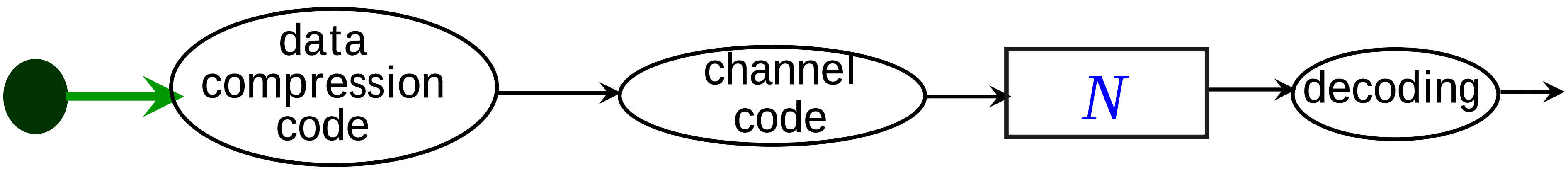
Result - III

Quantum Source Channel Separation Theorems

Theorem : It possible to transmit a quantum information source (ρ) reliably over a quantum channel (N) if & only if

von Neumann entropy $$H(\rho) \leq Q(N)$$ quantum capacity

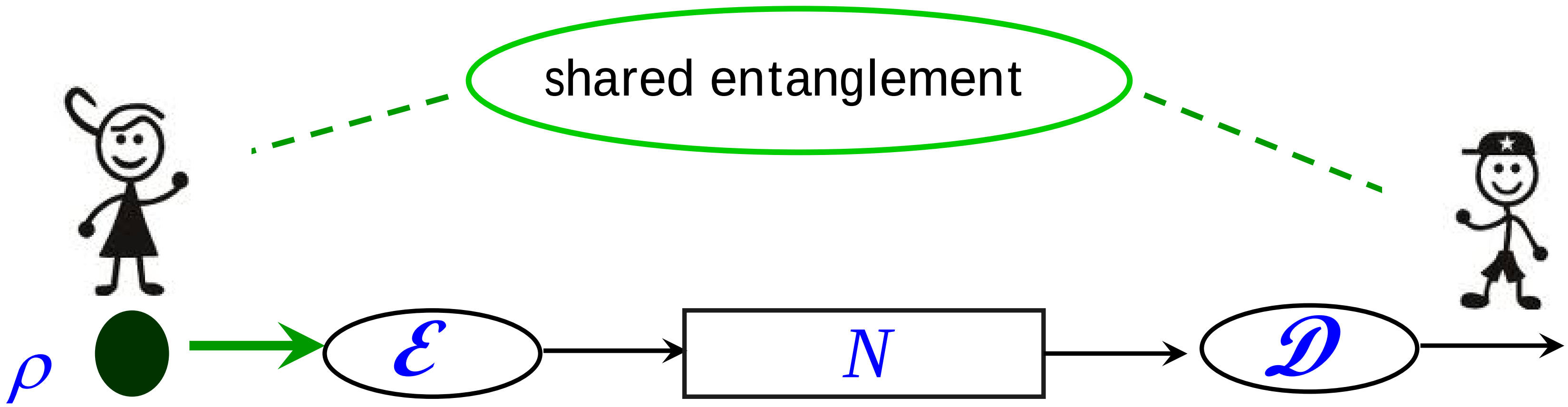
- **Implication:**



- a 2-stage encoding & decoding with the best data compression + quantum error correction codes is optimal!

What if $H(\rho) > Q(N)$?

- Suppose Alice & Bob share entanglement, i.e.,



- **Theorem:** it is possible to transmit the source ρ over the channel N, up to some distortion D

if & only if

$$R_q^{ea}(D) \leq \frac{1}{2} I_c(\rho, N)$$

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Summary

$R_q^{ea}(D):$
- entanglement-assisted quantum rate distortion function
- in terms of the quantum mutual information

$R_q(D):$
- unassisted quantum rate distortion function
- in terms of regularised entanglement of purification

- Quantum Source Channel Separation Theorems

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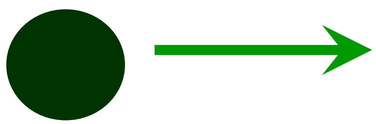
- *The following are some extra slides which I have removed/modified to make the talk 20 minute long.*

Shannon's Coding Theorems

Source Coding Theorem



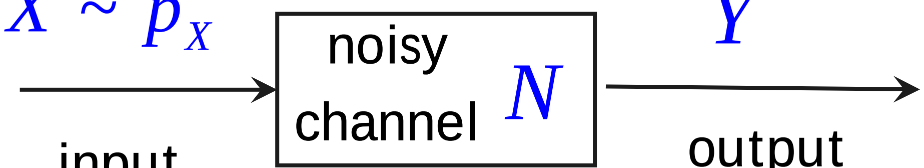
$$X \sim p_X \qquad ; \; x \in \mathrm{X}$$

The fundamental limit of data compression:

= *Shannon entropy of the source:*

$$H(X)$$

Channel Coding Theorem



The fundamental limit on the rate of reliable information transmission:

= *Capacity of the channel*

$$C(N) = \max_{p_X} I(X:Y)$$

mutual information

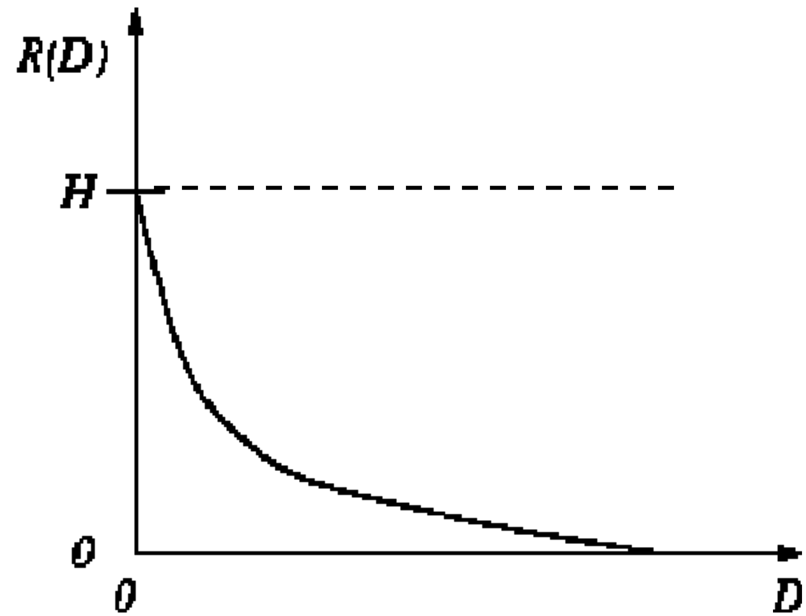
Rate Distortion Function

$$R(D) = \min_{\substack{p_{Y|X}: \\ \mathrm{E}(d(X,Y)) \le D}} I(X:Y)$$

- For $D = 0, \quad R(D) = H(X)$

  $Y = X$

- For $D > 0, \quad R(D) < H(X)$



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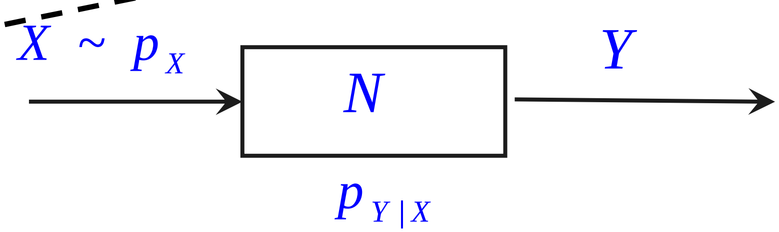
Barnum's Conjecture

Quantum Rate Distortion function

$$R_q(D) = \min_{\substack{N:\,\text{CPTP} \\ d(\rho,N)\le D}} I_c(\rho, N)$$

- Motivation : ← similarity in the classical case

Rate Distortion Function	Capacity of a noisy channel
$R(D) = \min_{\substack{p_{Y\mid X}(y\mid x): \\ E(d(x,y))\le D}} I(X:Y)$	$C(N) = \max_{p_X} I(X:Y)$



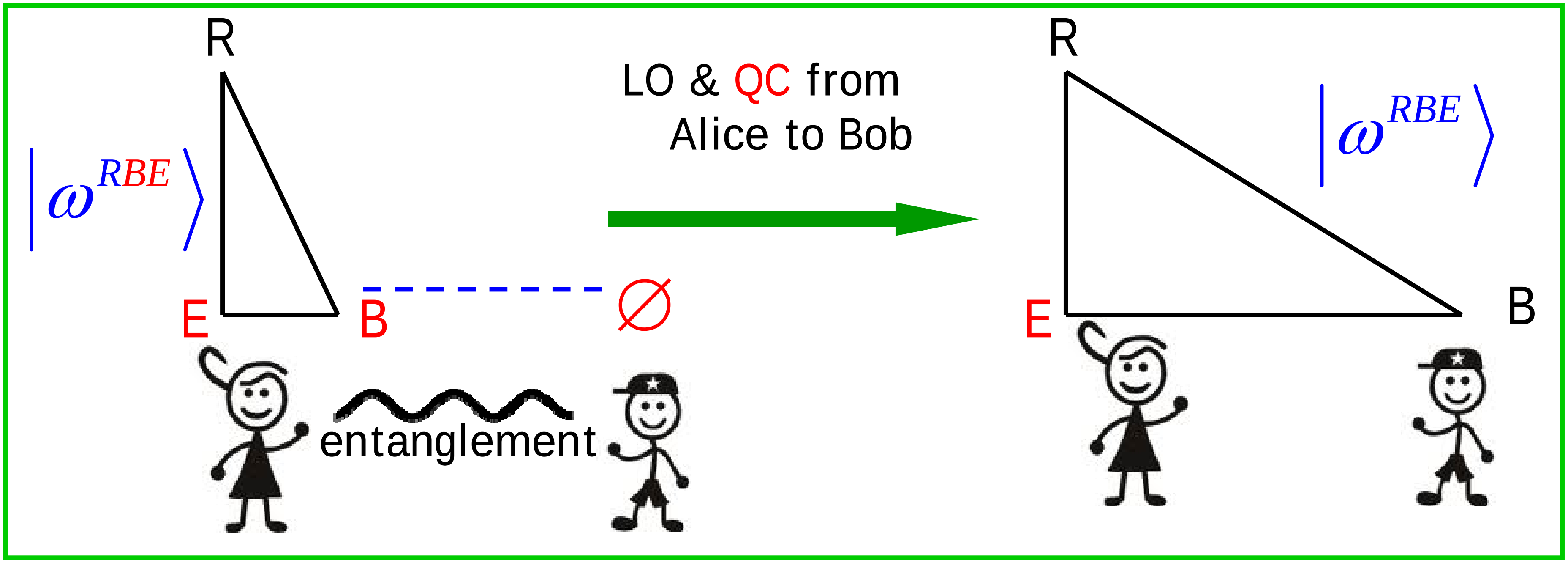
mutual information

Quantum capacity of a noisy quantum channel

$$Q(N) = \lim_{n\to\infty} \frac{1}{n} \max_{\rho} I_c\left(\rho, N^{\otimes n}\right)$$

coherent information

State Splitting



Resource Inequality : $\omega^{BE|\emptyset} + q[q \to q] + e[qq] \underset{\infty}{\geq} \omega^{E|B}$

$$e = \frac{1}{2} I(B:E)_\omega \,;$$

$$q = \frac{1}{2} I(R:B)_\omega = \min_{\substack{N:\ \mathrm{CPTP} \\ d(\rho, N) \leq D}} \frac{1}{2} I(R:B)_\omega$$

achievable rate !

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Result - II

- Unassisted Quantum rate distortion function:

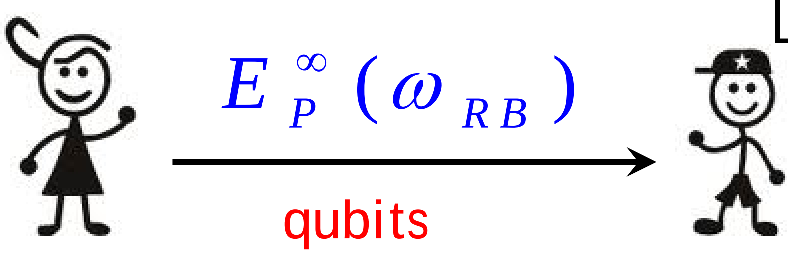
$$R_q(D) = \lim_{n\to\infty} \frac{1}{n} \min_{\substack{N:\,\mathrm{CPTP} \\ d(\rho, N^{(n)}) \le D}} E_P^{\infty}\left(N^{(n)}(\rho^{\otimes n})\right)$$

Channel simulation (QRST) in the **absence** of shared entanglement $\Longrightarrow$ achievability of this rate

To simulate output of $N^{\otimes n}$ when input is $\rho^{\otimes n}$

Resource Inequality :

$$E_P^{\infty}(\omega_{RB})[q \to q] \underset{\infty}{\geq} \omega_B$$



$$\omega_B = N(\rho)$$

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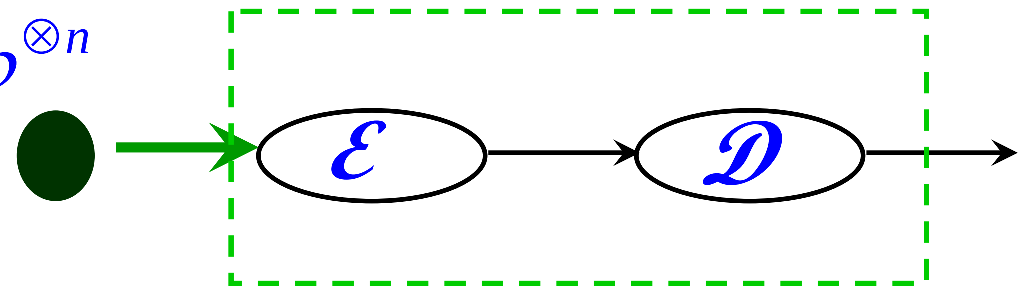
- Unassisted Quantum rate distortion function:

$$R_q(D) = \lim_{n\to\infty} \frac{1}{n} \min_{\substack{N:\ \text{CPTP} \\ d(\rho, N^{(n)}) \le D}} E_P^{\infty}\left(N^{(n)}(\rho^{\otimes n})\right)$$

- Why the double regularization?

- Storage setting:

n uses of the source $\rho^{\otimes n}$



$\mathcal{D} \circ \mathcal{C} \neq N^{\otimes n}$

- whereas in **QRST:** simulates $N^{\otimes n}$

- To prove "$=$" need to prove the converse
  - done using entropic inequalities