Advances in classical communication for network quantum information theory

Omar Fawzi, Patrick Hayden, Ivan Savov, Pranab Sen, Mark M. Wilde

School of Computer Science, McGill University

Tata Institute of Fundamental Research

Main papers: Sen '11 1109.0802, FHSSW'11 1102.2624

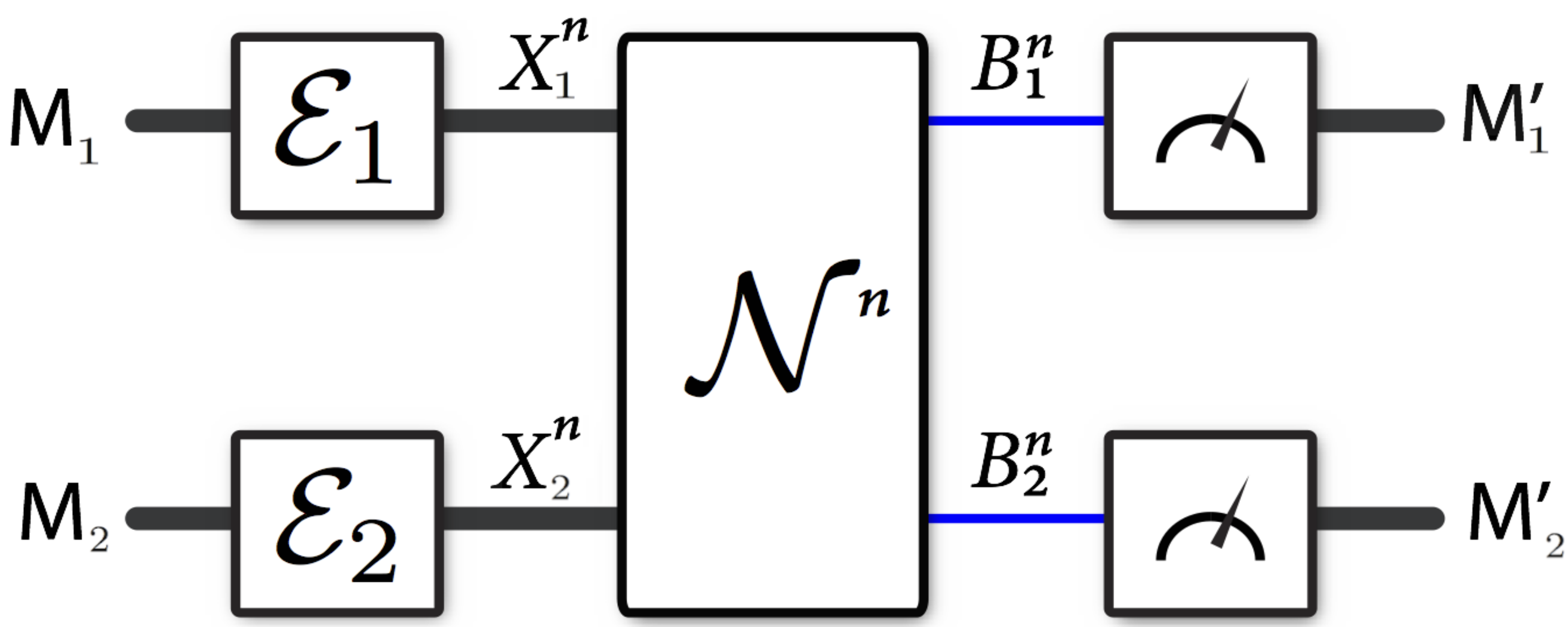
Others: 1102.2627, 1102.2955, 1107.1347, 1111.3645

Quantum Information Processing 2012, Montreal, Quebec, December 12, 2011

Interference Channel

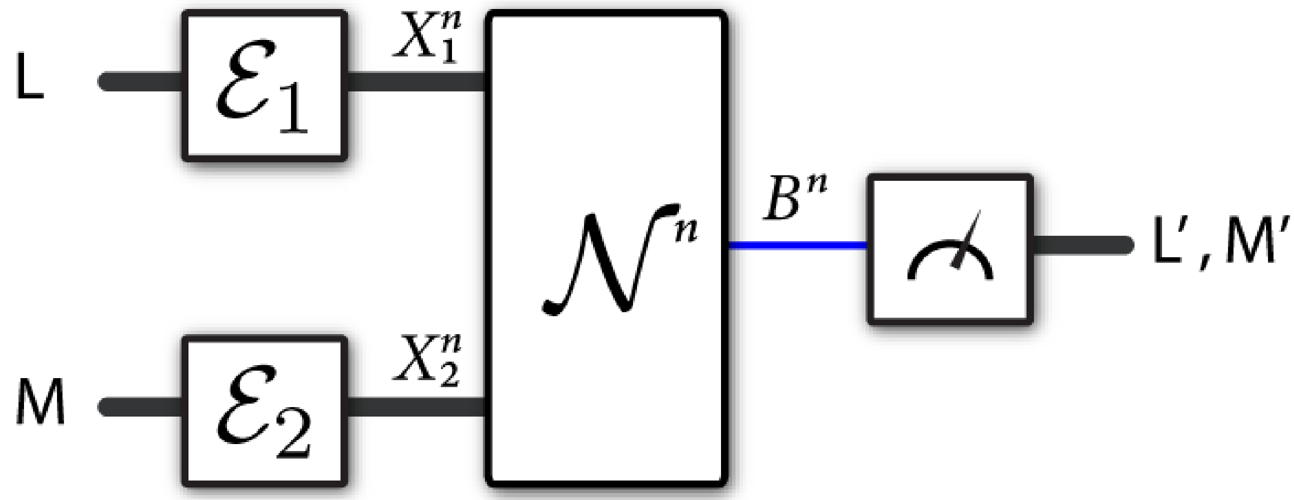
Quantum interference channel has two senders and two receivers

Model as a *ccqq* channel: $x_1, x_2 \longrightarrow \rho_{x_1,x_2}^{B_1,B_2}$



Task: First sender wants to communicate classical data to first receiver and second sender to second receiver

Multiple Access Channel (MAC)



Channel is ccq:

$$x_1, x_2 \longrightarrow \rho^B_{x_1,x_2}$$

Achievable rate region:

$$R_1 \leq I\left(X_1; B|X_2\right)_\rho,$$

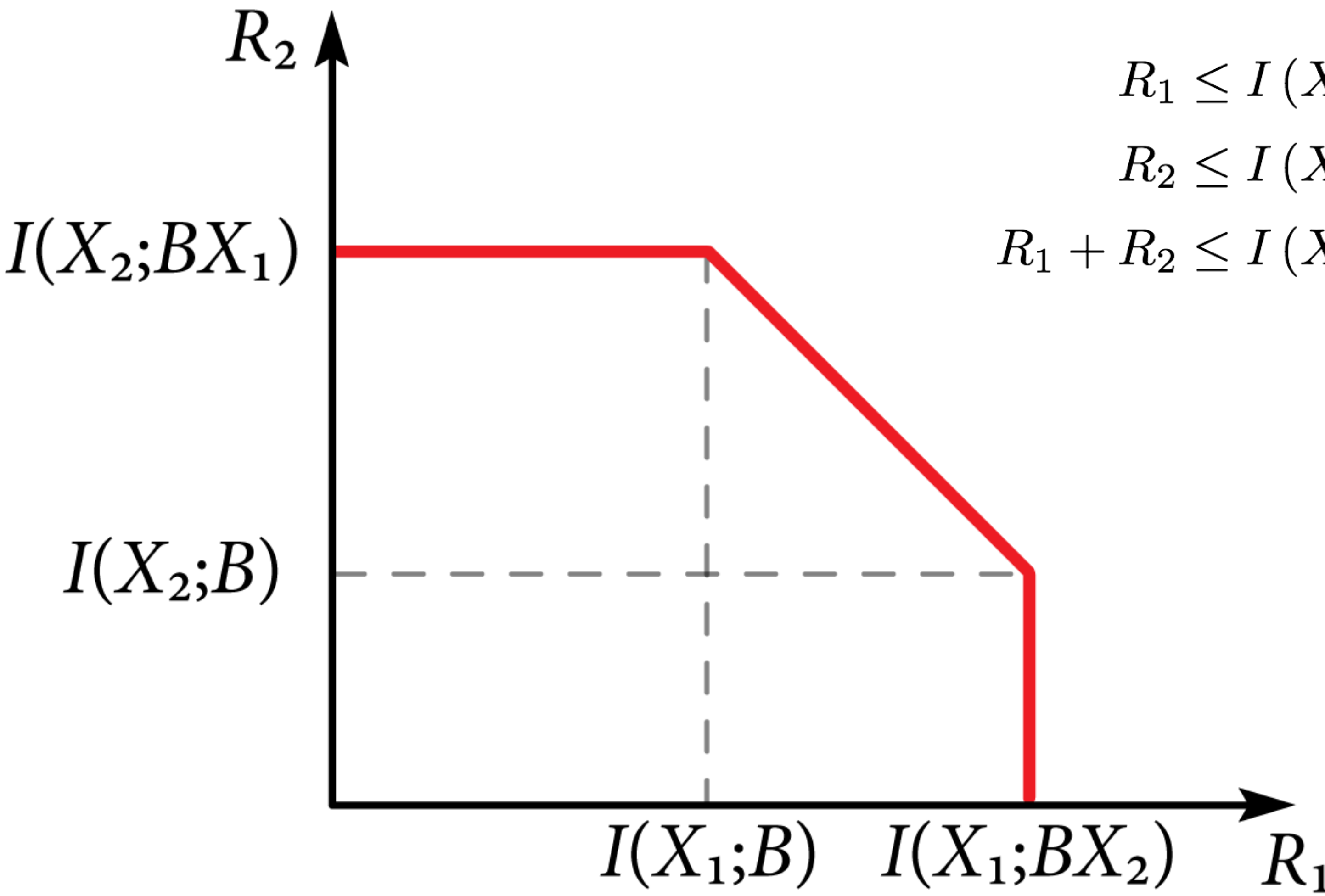
$$R_2 \leq I\left(X_2; B|X_1\right)_\rho,$$

$$R_1 + R_2 \leq I\left(X_1 X_2; B\right)_\rho$$

$$\rho^{X_1 X_2 B} \equiv \sum_{x_1, x_2} p\left(x_1\right) p\left(x_2\right) \left|x_1\right\rangle \left\langle x_1\right|^{X_1} \otimes \left|x_2\right\rangle \left\langle x_2\right|^{X_2} \otimes \rho^B_{x_1,x_2}.$$

A. Winter, "The Capacity of the Quantum Multiple Access Channel," arXiv:quant-ph/9807019

Achievable Rate Region for the MAC

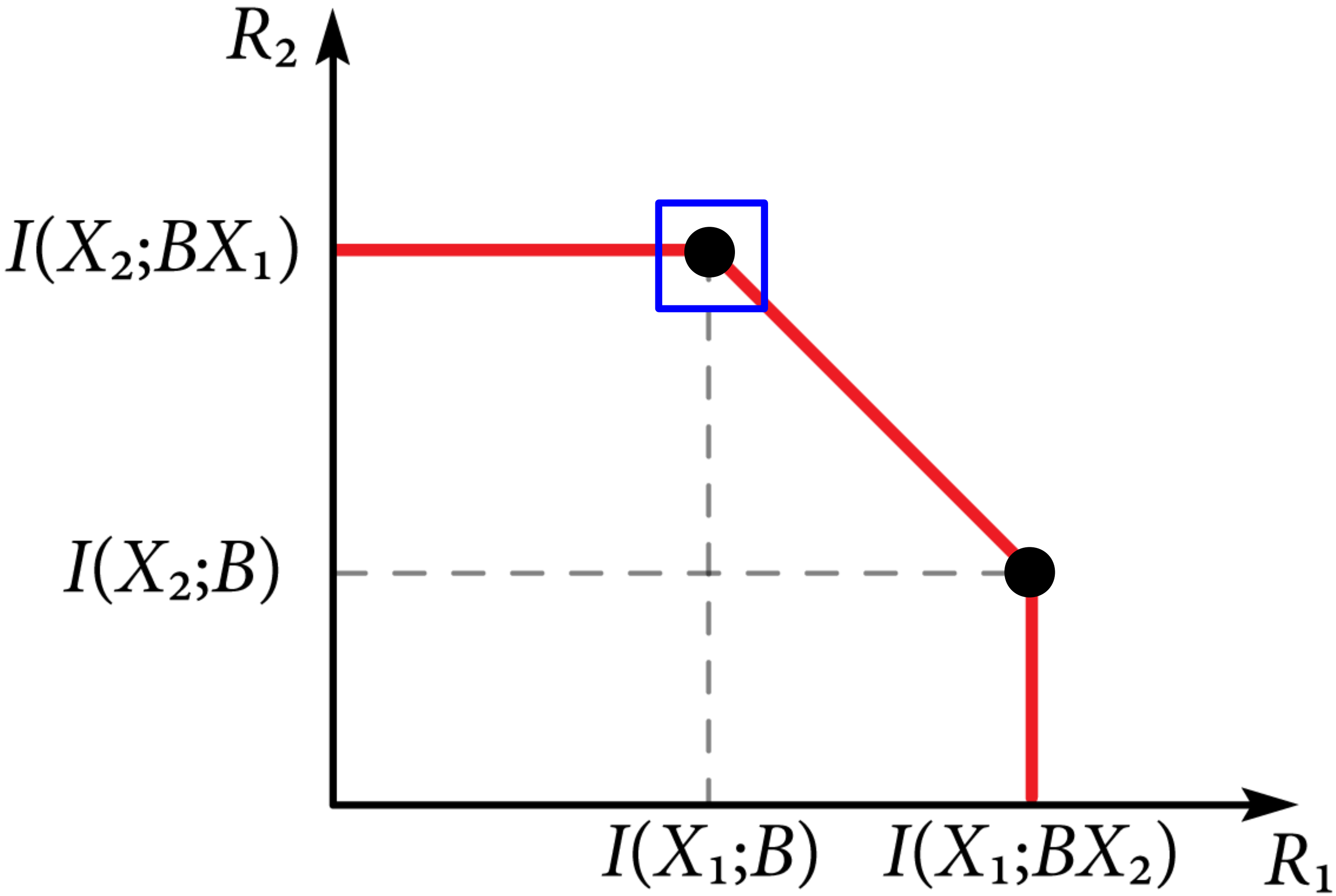


$$R_1 \leq I(X_1; B|X_2)_\rho ,$$
$$R_2 \leq I(X_2; B|X_1)_\rho ,$$
$$R_1 + R_2 \leq I(X_1 X_2; B)_\rho$$

Can achieve this rate region with either

1) **quantum successive decoding** (Winter '98)
2) **quantum simultaneous decoding** (Sen '11 and FHSSW '11)

Quantum Successive Decoding



Achieve the corner points of the above region and time-share!

For example, decode Sender 1 at rate $I(X_1;B)$.

Then decode Sender 2 at rate $I(X_2;BX_1)$.

We just need **point-to-point codes** for quantum successive decoding...

A. Winter, "The Capacity of the Quantum Multiple Access Channel," arXiv:quant-ph/9807019

Point-to-Point HSW Coding

Consider a channel
with **classical input** and **quantum output**:

$$x \longrightarrow \rho_x$$

If we use **random coding** according to $p_X(x)$ at input,
then **expected density operator** at output is

$$\rho \equiv \sum_x p_X(x) \rho_x$$

Can use **random coding** to select codebook $\{x^n(m)\}_{m}$
but need to design a **decoding POVM**

Sequential Decoding

Consider spectral decomposition: $\rho = \sum_z \lambda_z |z\rangle\langle z|$

Use **typical projector** for expected state:

$$\Pi^n_{\rho,\delta} \equiv \sum_{z^n \in T_\delta^{Z^n}} |z^n\rangle\langle z^n|$$

And **conditionally typical projectors** for codeword states:

$$\Pi_{\rho_{x^n},\delta} \equiv \bigotimes_x \Pi_{\rho_x,\delta}$$

Sequential decoding:
Just use conditionally typical projectors to ask,
"*Is it the first codeword? Is it the second codeword? ….*"

P. Sen, "Achieving the Han-Kobayashi inner bound ...", arXiv:1109.0802

Error Analysis

Analyze **error probability** for m[th] message:

$$1 - \text{Tr}\left\{\Pi_{\rho_m,\delta}\ \hat{\Pi}_{\rho_{m-1},\delta} \cdots \hat{\Pi}_{\rho_1,\delta}\ \rho_m\ \hat{\Pi}_{\rho_1,\delta} \cdots \hat{\Pi}_{\rho_{m-1},\delta}\ \Pi_{\rho_m,\delta}\right\},$$

where $\hat{\Pi} \equiv I - \Pi$

Upper bound this using a **noncommutative union bound**:

$$1 - \text{Tr}\{\Pi_N \cdots \Pi_1 \rho \Pi_1 \cdots \Pi_N\} \leq 2\sqrt{\sum_{i=1}^{N} \text{Tr}\{(I - \Pi_i)\rho\}}$$

1) Probability that codeword is **not typical**:

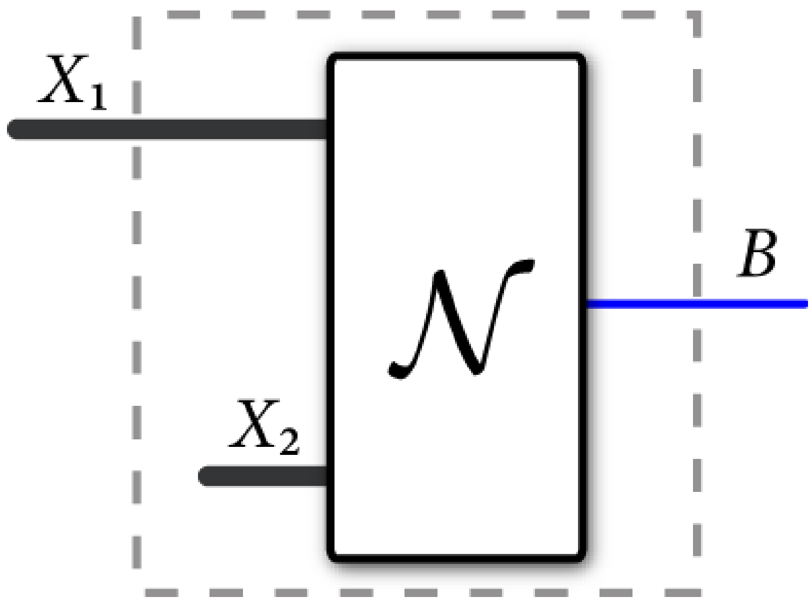
$$\text{Tr}\left\{(I - \Pi_{\rho_m,\delta})\rho_m\right\} \leq \epsilon$$

2) Probability that **some other codeword is typical**:

$$\mathbb{E}_{X^n}\left[\sum_{m' \neq m} \text{Tr}\left\{\Pi_{\rho_{m'}} \Pi_\rho \rho_m \Pi_\rho\right\}\right] \leq |\mathcal{M}|\, 2^{-nI(X;B)}$$

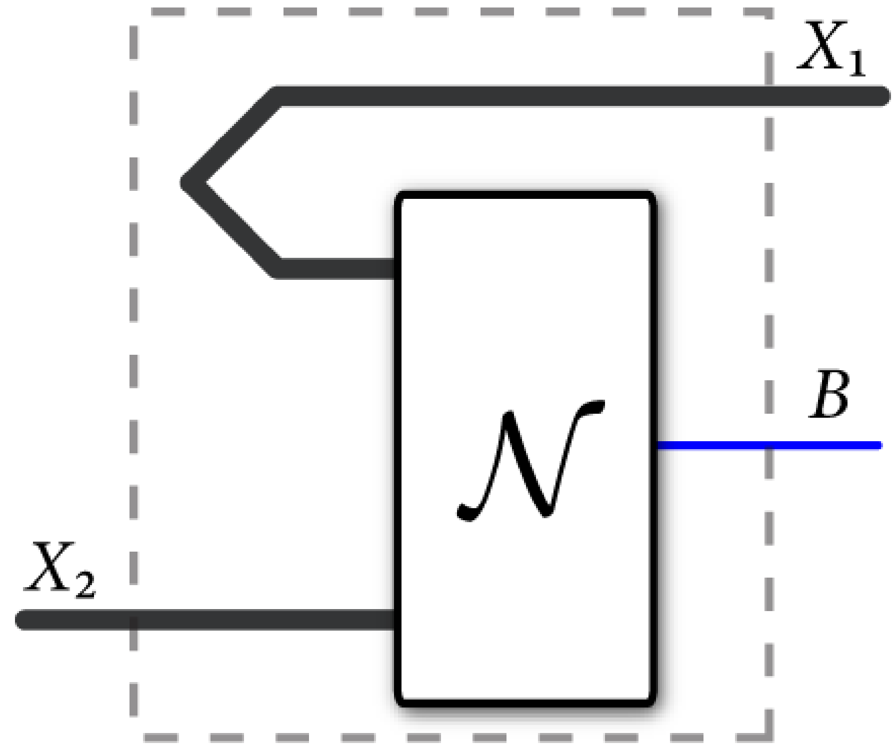
Back to Successive Decoding

Sender 1 and Receiver code for a point-to-point channel with Sender 2's input as **noise**



Can do so at rate $I(X_1;B)$

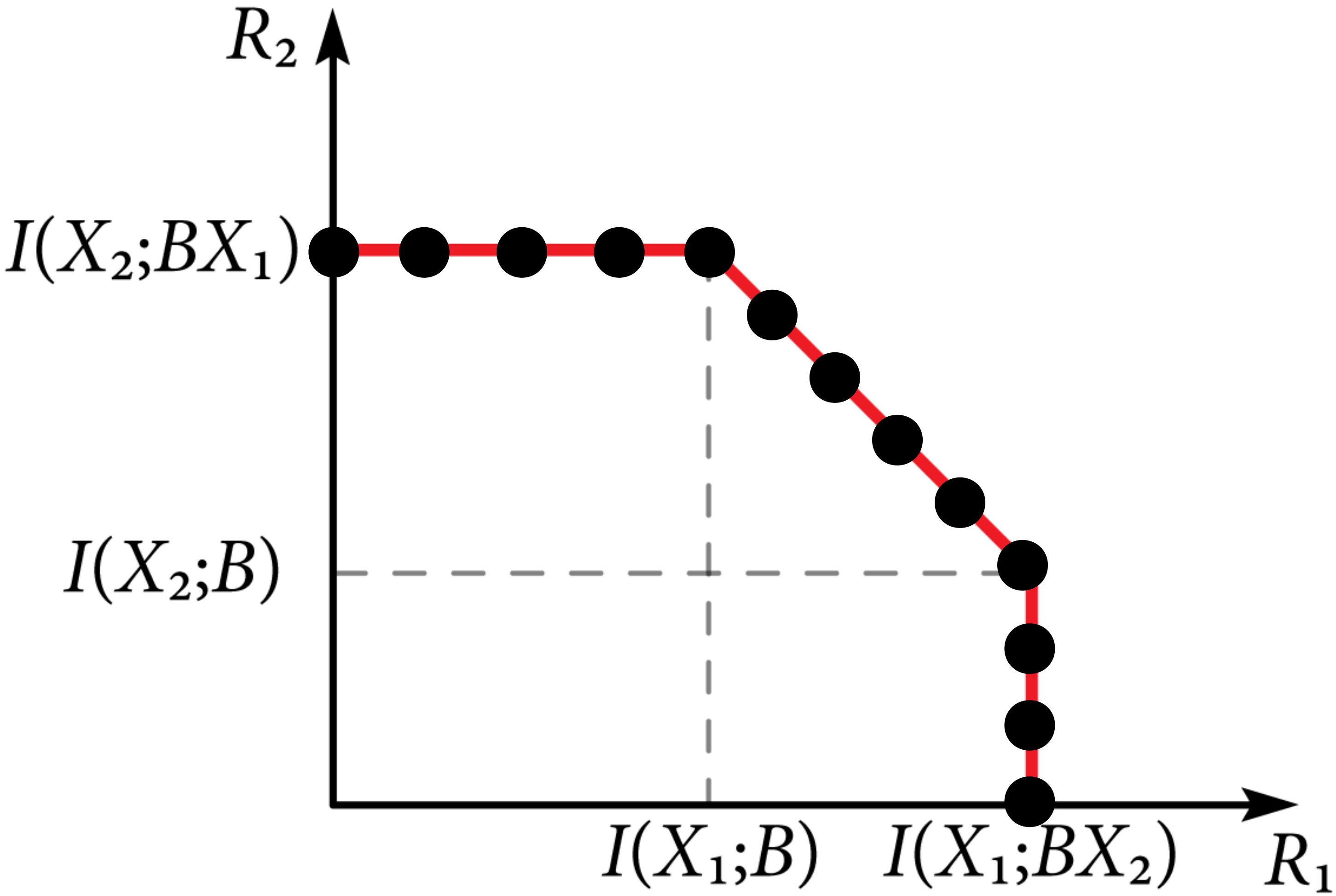
Sender 2 and Receiver code for a point-to-point channel with Sender 1's input **available**



Can do so at rate $I(X_2;BX_1)$

A. Winter, "The Capacity of the Quantum Multiple Access Channel," arXiv:quant-ph/9807019

Quantum Simultaneous Decoding



Achieve any point in the above region *without time-sharing*.

Use a novel decoding POVM that gets both messages at the same time

Has application in codes for quantum interference channel (and other channels)

P. Sen, "Achieving the Han-Kobayashi inner bound ...", 1109.0802
FHSSW, "Classical communication over a quantum interference channel, 1102.2624

Constructing a Quantum Simultaneous Decoder

Consider the following states:

$$\rho_{x_1,x_2}$$

$$\rho_{x_1} \equiv \sum_{x_2} p_{X_2}(x_2)\rho_{x_1,x_2}$$

$$\rho_{x_2} \equiv \sum_{x_1} p_{X_1}(x_1)\rho_{x_1,x_2}$$

$$\rho \equiv \sum_{x_1,x_2} p_{X_1}(x_1)p_{X_2}(x_2)\rho_{x_1,x_2}$$

n-fold versions of them have typical projections:

$$\Pi_{x_1^n,x_2^n} \equiv \Pi_{\rho_{x_1^n,x_2^n},\delta}$$

$$\Pi_{x_1^n} \equiv \Pi_{\rho_{x_1^n},\delta}$$

$$\Pi_{x_2^n} \equiv \Pi_{\rho_{x_2^n},\delta}$$

$$\Pi \equiv \Pi_{\rho,\delta}$$

Can use either the square-root measurement idea (FHSSW '11)

Or the sequential decoding idea (Sen '11)

P. Sen, "Achieving the Han-Kobayashi inner bound ...", 1109.0802
FHSSW, "Classical communication over a quantum interference channel, 1102.2624

Sequential Decoding Approach

The decoder seems to require the projectors

$$\Pi, \quad \Pi_{x_1^n}, \quad \Pi_{x_2^n}, \quad \Pi_{x_1^n, x_2^n}$$

in order to have good bounds on eigenvalues of averaged density operators

So we might consider making measurements from positive operators such as

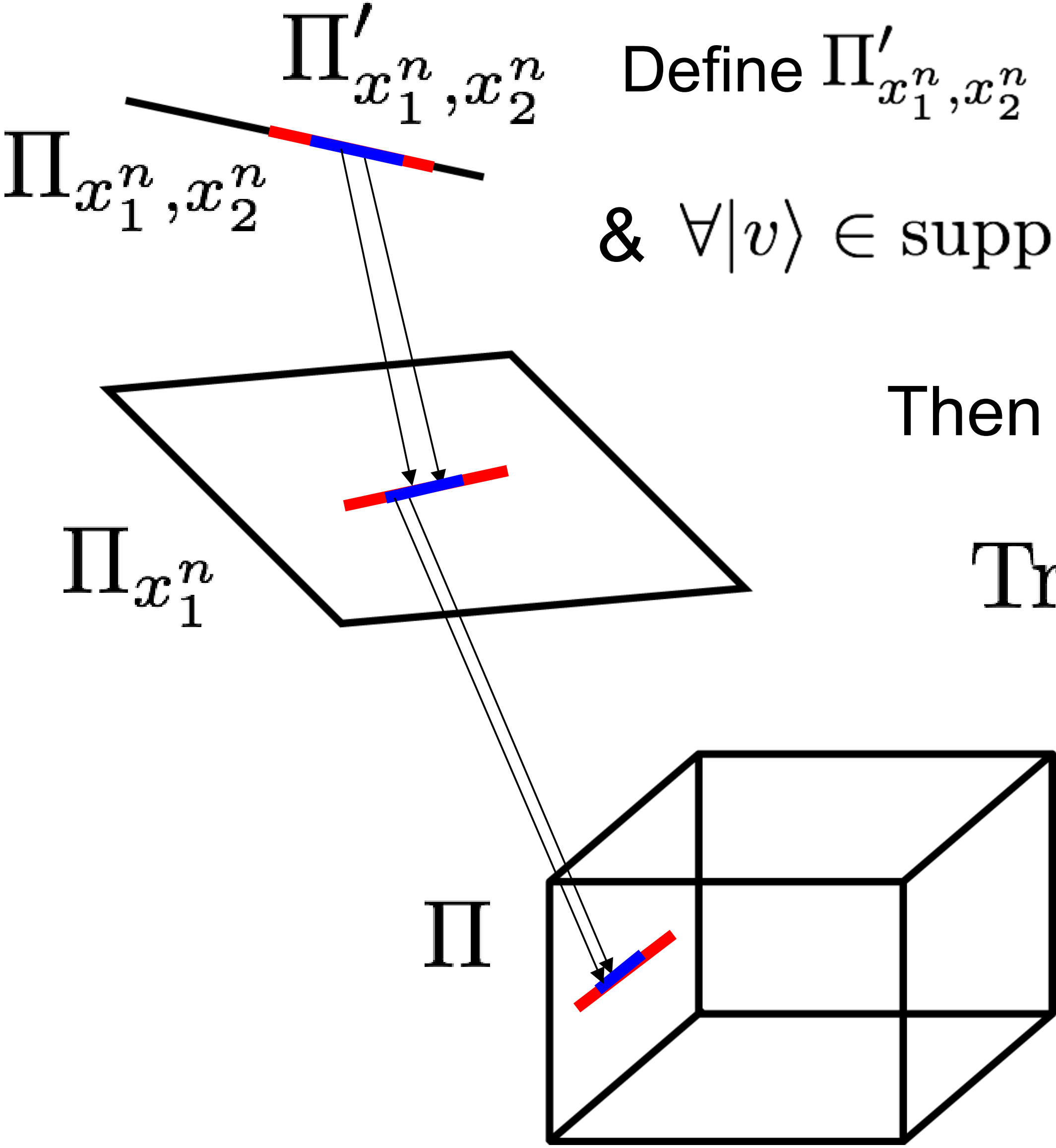
$$\Pi \, \Pi_{x_1^n} \, \Pi_{x_1^n, x_2^n} \, \Pi_{x_1^n} \, \Pi$$

But the noncommutative union bound requires projectors!

How to make an “intersection” projector?

P. Sen, “Achieving the Han-Kobayashi inner bound ...”, 1109.0802

"Intersection" of Projectors



Define $\Pi'_{x_1^n,x_2^n}$ by taking O.N. basis {|v>} for $\Pi_{x_1^n,x_2^n}$

& $\forall |v\rangle \in \mathrm{supp}(\Pi'_{x_1^n,x_2^n}) : \|\Pi\, \Pi_{x_1^n} |v\rangle\|_2 \geq 1 - \epsilon$

Then $\Pi'_{x_1^n,x_2^n} \leq \Pi_{x_1^n,x_2^n}$ and

$$\mathrm{Tr}\{\Pi'_{x_1^n,x_2^n} \rho_{x_1^n,x_2^n}\} \geq 1 - \epsilon$$

Define $\tilde{\Pi}_{x_1^n,x_2^n}$ by

$\mathrm{span}\{\Pi\, \Pi_{x_1^n} |v\rangle : |v\rangle \in \mathrm{supp}(\Pi'_{x_1^n,x_2^n})\}$

$\tilde{\Pi}_{x_1^n,x_2^n}$ plays the role of an

"intersection projector"

P. Sen, "Achieving the Han-Kobayashi inner bound ...", 1109.0802

Properties of “Intersection” Projectors

Two critical properties for error analysis:

Operator inequality that follows from construction:

$$\tilde{\Pi}_{x_1^n, x_2^n} \leq \Pi \, \Pi_{x_1^n} \, \Pi_{x_1^n, x_2^n} \, \Pi_{x_1^n} \, \Pi$$

We also have that

$$\mathrm{Tr}\{\tilde{\Pi}_{x_1^n, x_2^n} \, \rho_{x_1^n, x_2^n}\} \geq 1 - \epsilon$$

P. Sen, “Achieving the Han-Kobayashi inner bound ...”, 1109.0802

Simultaneous Decoder for the MAC

Just ask, "*Is it the first message pair? Is it the second message pair?*" etc. ...

The error in doing so is

$$1 - \text{Tr}\left\{\tilde{\Pi}_{\rho_{l,m},\delta}\ \hat{\tilde{\Pi}}_{\rho_{l-1,m},\delta} \cdots \hat{\tilde{\Pi}}_{\rho_{1,1},\delta}\ \rho_{l,m}\ \hat{\tilde{\Pi}}_{\rho_{1,1},\delta} \cdots \hat{\tilde{\Pi}}_{\rho_{l-1,m},\delta}\ \tilde{\Pi}_{\rho_{l,m},\delta}\right\}$$

Can "*smooth*" the state with the projector $\Pi_{x_2^n}$

Apply the **non-comm. union bound** to the above error probability.

P. Sen, "Achieving the Han-Kobayashi inner bound ...", 1109.0802

Simultaneous Decoder for the MAC (ctd.)

This leads to four error terms which we bound as

1) Pr{Not typical}

$$\mathrm{Tr}\left\{(I - \tilde{\Pi}_{\rho_{l,m},\delta})\Pi_{\rho_m,\delta}\rho_{l,m}\Pi_{\rho_m,\delta}\right\} \leq \epsilon'$$

2) Pr{1st message wrong}

$$\mathbb{E}_{X_1^n,X_2^n} \sum_{l' \neq l} \mathrm{Tr}\left\{\tilde{\Pi}_{\rho_{l',m}}\Pi_{\rho_m}\rho_{l,m}\Pi_{\rho_m}\right\} \leq |\mathcal{L}|\, 2^{-nI(X_1;BX_2)}$$

3) Pr{2nd message wrong}

$$\mathbb{E}_{X_1^n,X_2^n} \sum_{m' \neq m} \mathrm{Tr}\left\{\tilde{\Pi}_{\rho_{l,m'}}\Pi_{\rho_m}\rho_{l,m}\Pi_{\rho_m}\right\} \leq |\mathcal{M}|\, 2^{-nI(X_2;BX_1)}$$

Note: Very similar error analysis in FHSSW11 with square-root measurement

P. Sen, "Achieving the Han-Kobayashi inner bound ...", 1109.0802

Conclusion and Current Work

Most important open problem:

Find a general three-sender quantum simultaneous decoder

If true, would allow us to produce a more complete theory of information transmission over quantum network channels

Intuition for truth: The two-sender decoder given here shows that codewords are asymptotically distinguishable and thus are approximately orthogonal in the limit. Why wouldn't this be true for the three-sender case?

The techniques discussed here have further applications to

1) Entanglement-assisted comm. (Xu and Wilde 1107.1347)

2) Quantum broadcast channels (Savov and Wilde 1111.3645)

3) Quantum relay channels (Savov, Wilde, and Vu 1201.????)

4) Any other scenarios that make use of the two-sender simultaneous decoding tricks?

Noncommutative Union Bound

Noncommutative union bound:

$$1 - \mathrm{Tr}\{\Pi_N \cdots \Pi_1 \rho \Pi_1 \cdots \Pi_N\} \leq 2\sqrt{\sum_{i=1}^{N} \mathrm{Tr}\{(I - \Pi_i)\rho\}}$$

Consider similarity with union bound:

$$\Pr\{(A_1 \cap \cdots \cap A_N)^c\} = \Pr\{A_1^c \cup \cdots \cup A_N^c\} \leq \sum_{i=1}^{N} \Pr\{A_i^c\}$$

Should find **widespread application** in quantum info. theory

Already used in

plicit polar codes for classical data transmission

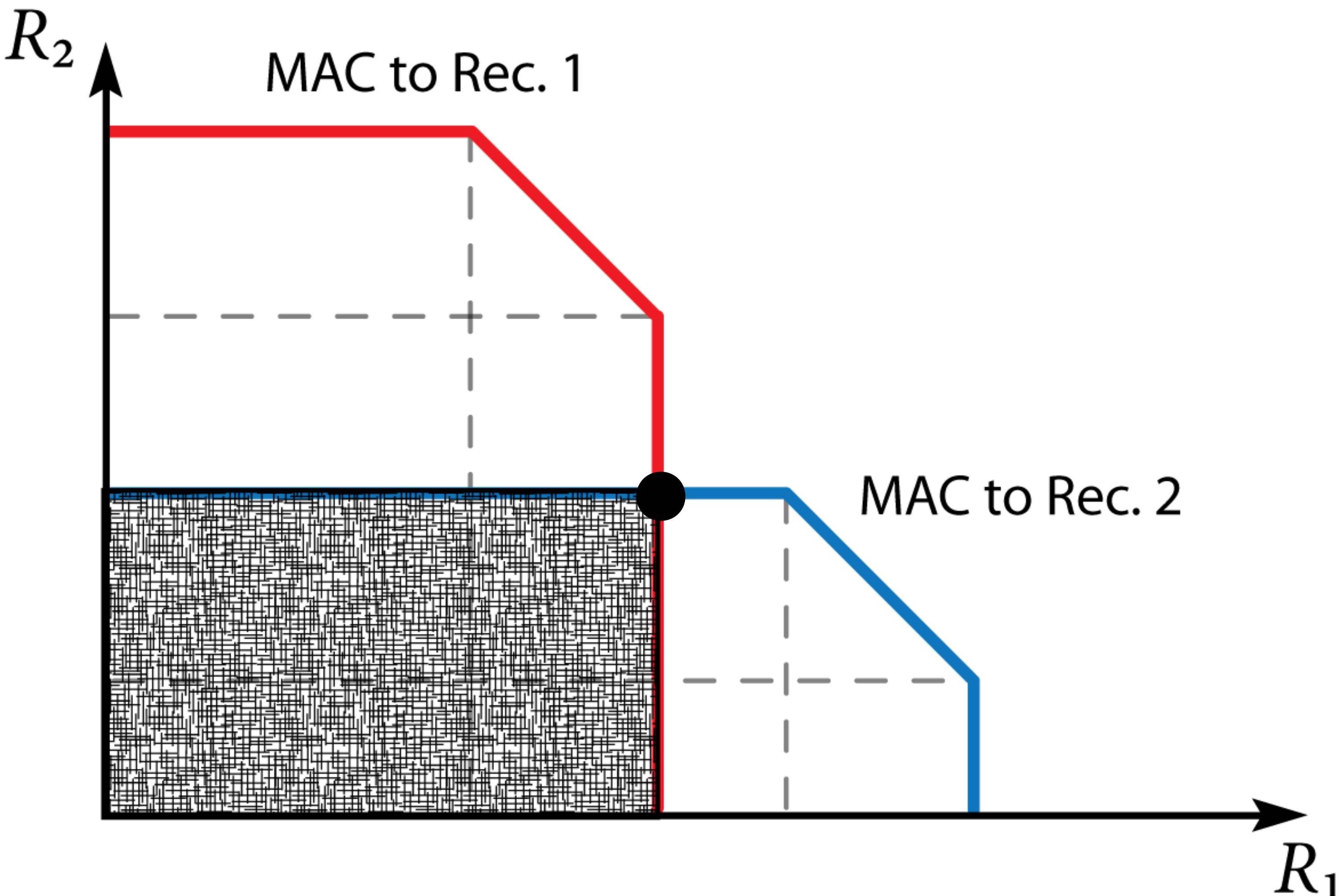
(Wilde & Guha 1109

detector for achieving classical capacity of bosonic channel

(Wilde, Guha, Lloyc

P. Sen, "Achieving the Han-Kobayashi inner bound ...", arXiv:1109.0802

Very Strong Interference

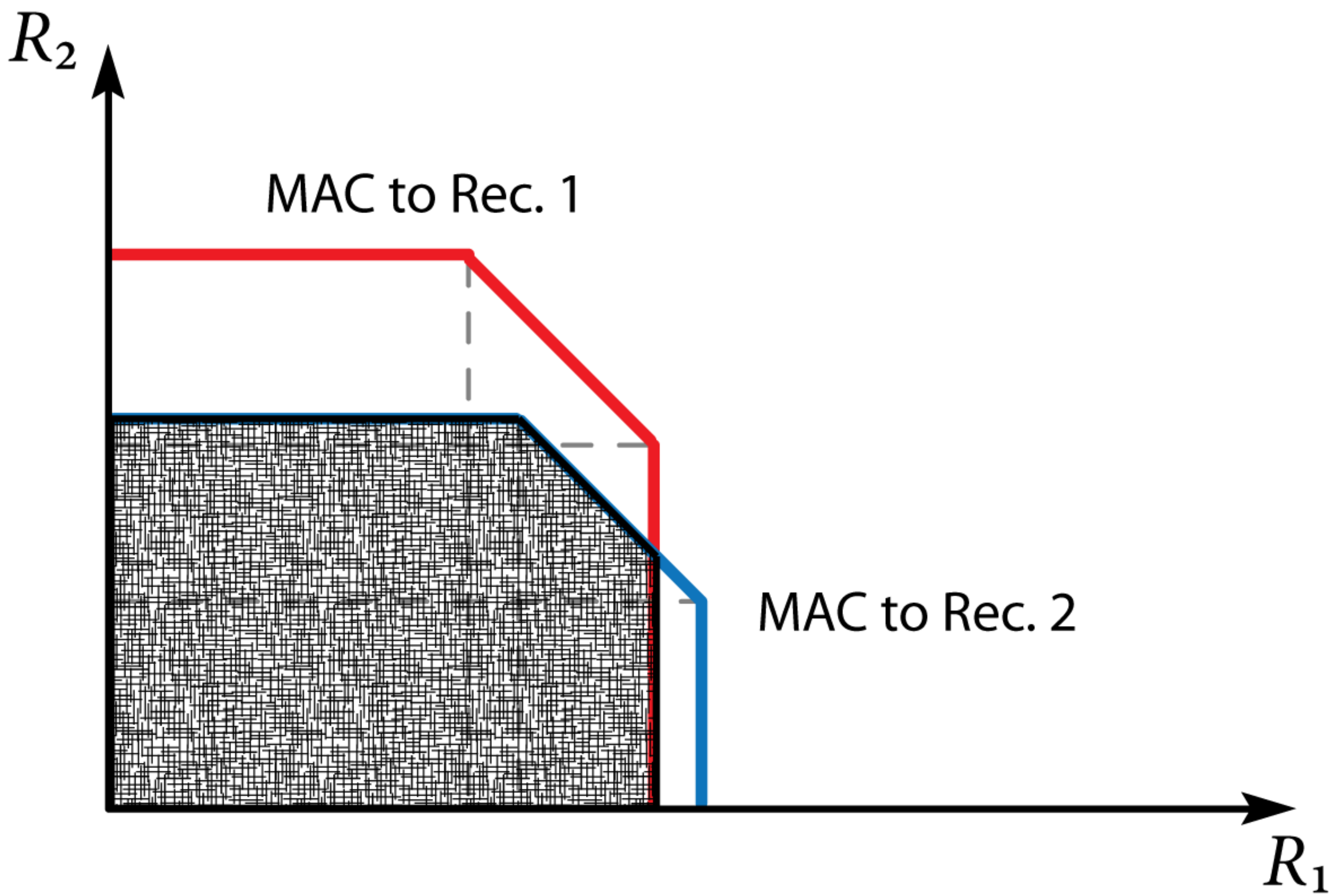


Conditions for VSI: $I(X_2;B_2X_1) \leq I(X_2;B_1)$ AND $I(X_1;B_1X_2) \leq I(X_1;B_2)$

Strategy: Interference is so "very strong" that it is best to decode "other sender" first

FHSSW, "Classical communication over a quantum interference channel, 1102.2624

Strong Interference



Conditions for SI: $I(X_2;B_2X_1) \leq I(X_2;B_1X_1)$ AND $I(X_1;B_1X_2) \leq I(X_1;B_2X_1)$

Each receiver uses a **two-sender quantum simultaneous decoder** to recover messages of both senders

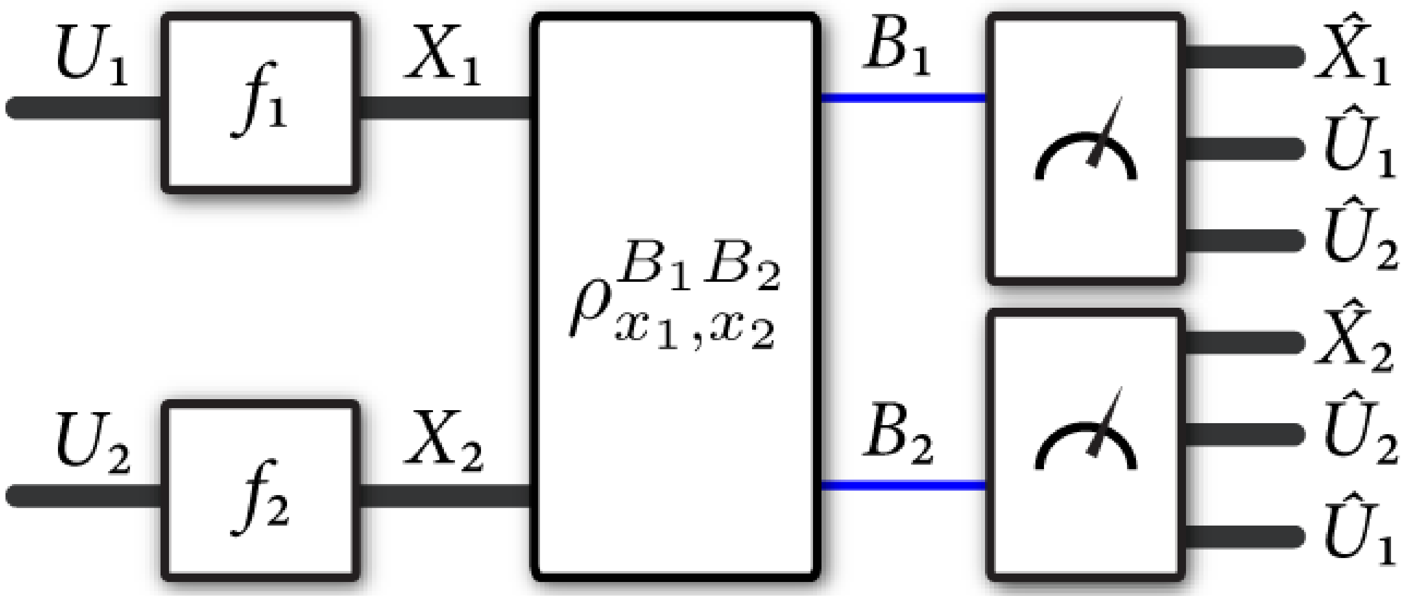
FHSSW, "Classical communication over a quantum interference channel, 1102.2624

General Interference Channel

Best achievable rate regions in classical case are due to Han and Kobayashi '81 and Chong, Motani, and Garg '06

Later was shown that the two rate regions are equivalent

One of us has shown how to achieve the CMG region with a **quantum simultaneous decoder** for a restricted 3-sender MAC and a geometric trick



P. Sen, "Achieving the Han-Kobayashi inner bound ...", 1109.0802