Minimal-Memory, Non-Catastrophic Quantum Convolutional Encoders

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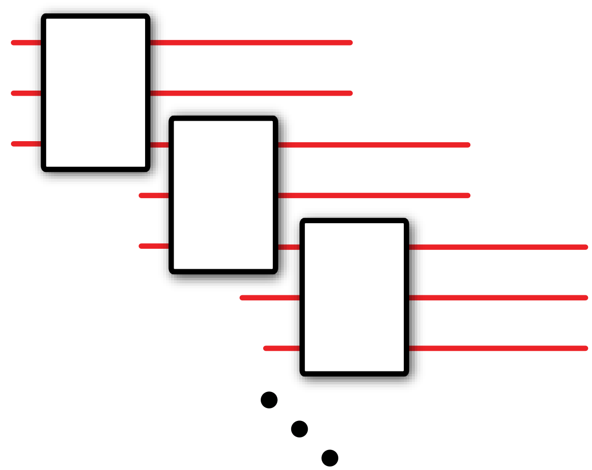


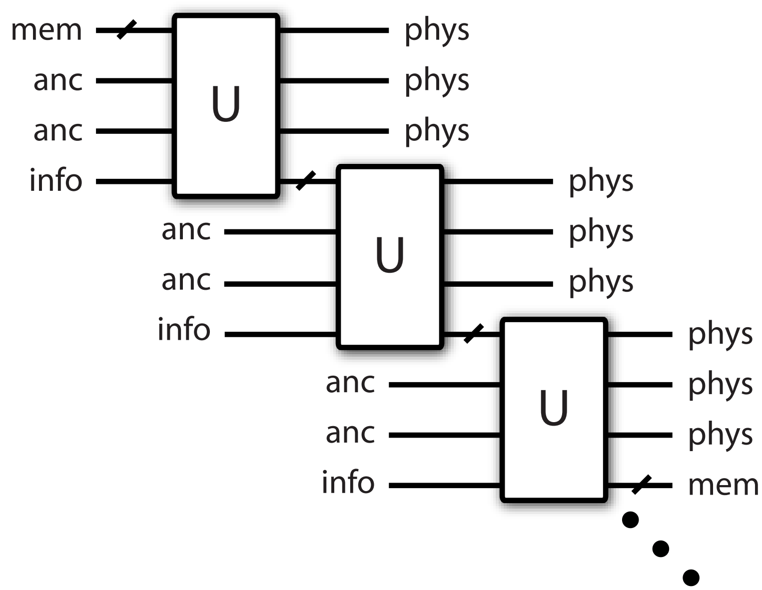
ISIT 2011, St. Petersburg, Russia

Quantum Convolutional Codes

A **quantum convolutional code** has a mathematical description with a set of stabilizer operators. An example code is below:

$$\begin{array}{ccc|ccc|ccc|c} X & X & X & X & Z & Y & I & I & I & \\ Z & Z & Z & Z & Y & X & I & I & I & \cdots \end{array}$$

A **quantum convolutional code** also requires an encoder—it would be ideal for this encoder to be **minimal-memory**



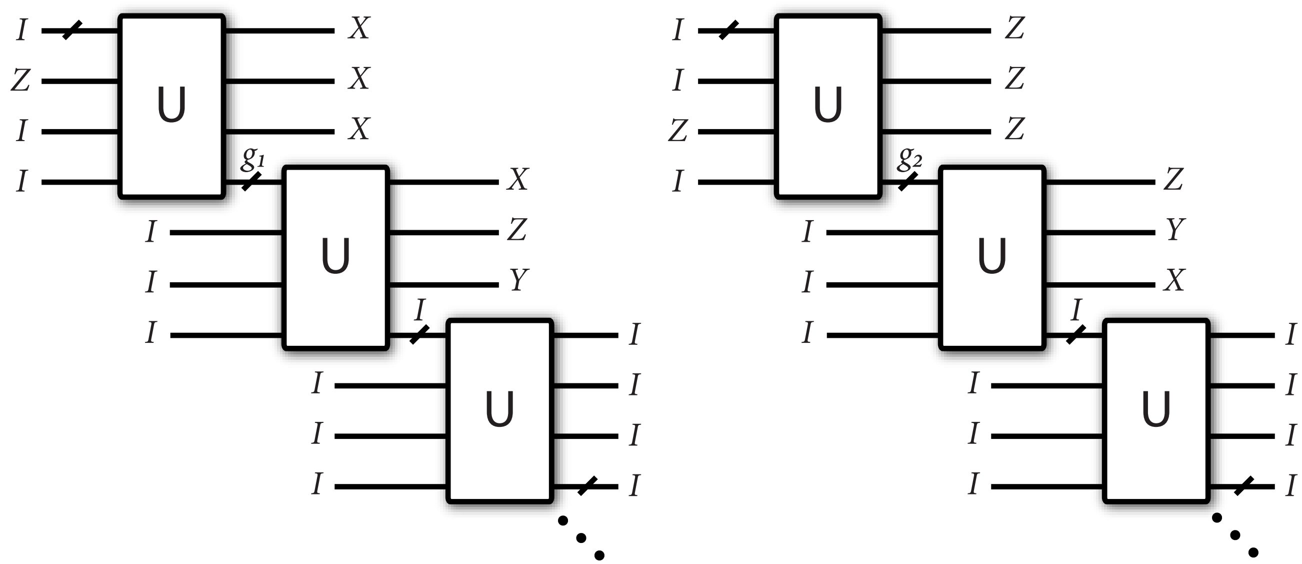
Ollivier and Tillich. arXiv:quant-ph/0304189
Forney, Grassl, Guha. arXiv:quant-ph/0511016
Grassl and Roetteler. arXiv:quant-ph/0602129

Finding a Minimal-Memory Encoder

The encoder transforms **unencoded operators** to **encoded operators**:

$$\begin{array}{c|ccc|ccc|c} I^{\otimes m} & Z & I & I & I & I & I & \\ I^{\otimes m} & I & Z & I & I & I & I & \end{array} \cdots \rightarrow \begin{array}{ccc|ccc|ccc|c} X & X & X & X & Z & Y & I & I & I & \\ Z & Z & Z & Z & X & Y & I & I & I & \end{array} \cdots$$

We can depict this transformation graphically:



So the encoder *U* should act as follows:

$$\begin{array}{cccc} I^{\otimes m} & Z & I & I \\ I^{\otimes m} & I & Z & I \\ g_1 & I & I & I \\ g_2 & I & I & I \end{array} \rightarrow \begin{array}{cccc} X & X & X & g_1 \\ Z & Z & Z & g_2 \\ X & Z & Y & I^{\otimes m} \\ Z & Y & X & I^{\otimes m} \end{array}$$

Wilde, Houshmand, and Hosseini-Khayat. arXiv:1011.5535

Finding a Min-Memory Encoder (Ctd.)

The memory Pauli operators g_1 and g_2
should respect the input-output commutation relations:

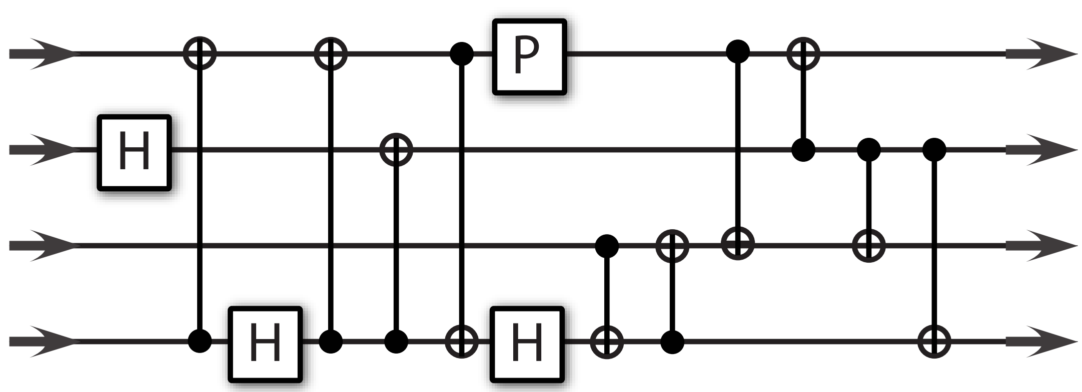
$$\{g_1, g_2\} = 0$$

A good choice for these operators is then $g_1 = X$ and $g_2 = Z$.

The encoder acts as follows and requires **only one memory qubit!**

$$\begin{array}{cccc} I & Z & I & I \\ I & I & Z & I \\ X & I & I & I \\ Z & I & I & I \end{array} \quad \rightarrow \quad \begin{array}{cccc} X & X & X & X \\ Z & Z & Z & Z \\ X & Z & Y & I \\ Z & Y & X & I \end{array}$$

One realization of the encoder is below:



Wilde, Houshmand, and Hosseini-Khayat. arXiv:1011.5535

Non-Catastrophicity

An encoder is **non-catastrophic** if *all zero physical-weight cycles have zero logical weight*.

This property is desirable because errors do not propagate infinitely for such encoders.

The minimal-memory encoder from the previous slide is **non-catastrophic**. Why?

Suppose (for a contradiction) that the encoder is **catastrophic**.
That is, there is some cycle of the following form:

$$\begin{matrix} h_1 & s_{1,1} & s_{1,2} & l_1 \\ \vdots & \vdots & \vdots & \vdots \\ h_p & s_{p,1} & s_{p,2} & l_p \end{matrix} \quad \longrightarrow \quad \begin{matrix} I & I & I & h_2 \\ \vdots & \vdots & \vdots & \vdots \\ I & I & I & h_1 \end{matrix}$$

Then, the one-qubit memory states of this cycle should commute with both *X* and *Z*.

But this can only happen if the memory states are equal to the identity!

Thus, the encoder is non-catastrophic...

Wilde, Houshmand, and Hosseini-Khayat. arXiv:1011.5535

Finding an Online Decoder

Would like to find an **online decoder** for the encoder found from before

The encoder transforms the **unencoded logical operators** as follows:

$$\begin{array}{c|ccc|ccc|c} I^{\otimes m} & I & I & X & I & I & I \\ I^{\otimes m} & I & I & Z & I & I & I \end{array} \cdots \rightarrow \begin{array}{ccc|ccc|ccc|c} Z & Y & I & X & Z & Y & I & I & I \\ Y & I & Z & X & Z & Y & I & I & I \end{array} \cdots$$

Need to decode these in addition to decoding the encoded stabilizer operators

Can use the same technique!

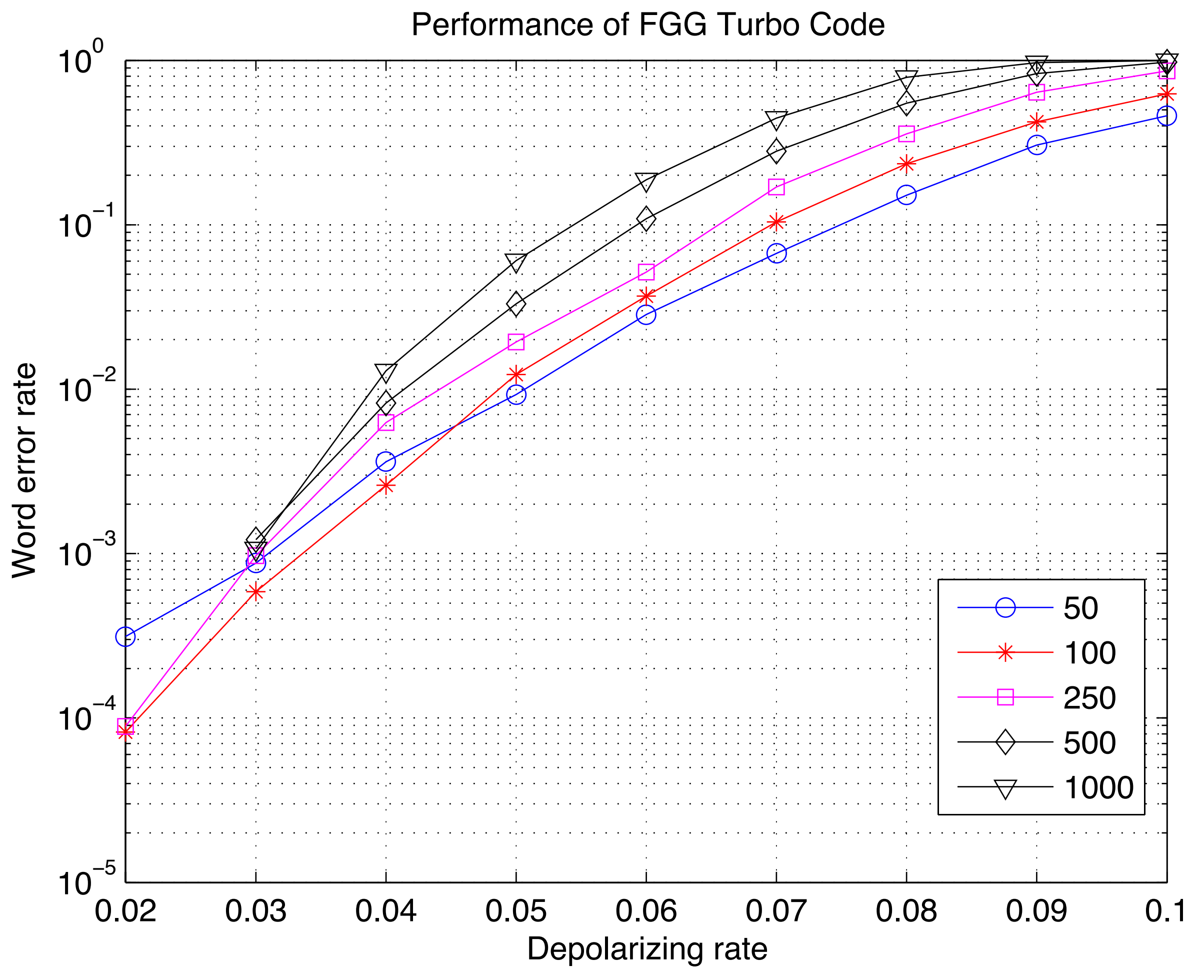
$$\begin{array}{cccc} I^{\otimes m} & Z & Y & I \\ I^{\otimes m} & Y & I & Z \\ I^{\otimes m} & X & X & X \\ I^{\otimes m} & Z & Z & Z \\ g_1' & X & Z & Y \\ g_2' & X & Z & Y \\ g_3' & X & Z & Y \\ g_4' & Z & Y & X \end{array} \rightarrow \begin{array}{cccc} I & I & I & g_1' \\ I & I & I & g_2' \\ I & I & I & g_3' \\ I & I & I & g_4' \\ I & I & X & I^{\otimes m} \\ I & I & Z & I^{\otimes m} \\ Z & I & I & I^{\otimes m} \\ I & Z & I & I^{\otimes m} \end{array}$$

Online decoder requires two memory qubits and is non-catastrophic

Wilde, Houshmand, and Hosseini-Khayat. arXiv:1011.5535

Simulation Results

The encoder for the FGG code requires only one memory qubit, and so we can simulate its performance in a quantum turbo code.



The peformance curve for this quantum turbo code does not feature a pseudothreshold, as is the case for other quantum turbo codes.

Poulin, Tillich, and Ollivier. arXiv:0712.2888

Wilde, Houshmand, and Hosseini-Khayat. arXiv:1011.5535

More General Example

How to handle **more general** examples?

$$
\begin{array}{rcl|cccc|cccc|cccc}
h_1 & = & X\ X\ X\ X & X & X & I & X & I & X & I & I & I & I & X & X \\
h_2 & = & Z\ Z\ Z\ Z & Z & Z & I & Z & I & Z & I & I & I & I & Z & Z
\end{array}
$$

Similar idea for encoder:

Mem.	Anc.		Info.	
$I^{\otimes m}$	Z	I	I	I
$g_{1,1}$	I	I	I	I
$g_{1,2}$	I	I	I	I
$g_{1,3}$	I	I	I	I
$I^{\otimes m}$	I	Z	I	I
$g_{2,1}$	I	I	I	I
$g_{2,2}$	I	I	I	I
$g_{2,3}$	I	I	I	I

$\rightarrow$

Phys.				Mem.
X	X	X	X	$g_{1,1}$
X	X	I	X	$g_{1,2}$
I	X	I	I	$g_{1,3}$
I	I	X	X	$I^{\otimes m}$
Z	Z	Z	Z	$g_{2,1}$
Z	Z	I	Z	$g_{2,2}$
I	Z	I	I	$g_{2,3}$
I	I	Z	Z	$I^{\otimes m}$

Need to consider commutation relations again...

Houshmand, Hosseini-Khayat, and Wilde. arXiv:1105.0649

More General Example (Ctd.)

Enumerate **all of the commutation relations** for memory operators:

$$[g_{1,1}, g_{1,2}] = [g_{1,1}, g_{1,3}] = [g_{1,1}, g_{2,1}] = \{g_{1,1}, g_{2,2}\} = \{g_{1,1}, g_{2,3}\} = 0,$$
$$[g_{1,2}, g_{1,3}] = \{g_{1,2}, g_{2,1}\} = \{g_{1,2}, g_{2,2}\} = [g_{1,2}, g_{2,3}] = 0,$$
$$\{g_{1,3}, g_{2,1}\} = [g_{1,3}, g_{2,2}] = [g_{1,3}, g_{2,3}] = 0,$$
$$[g_{2,1}, g_{2,2}] = [g_{2,1}, g_{2,3}] = 0,$$
$$[g_{2,2}, g_{2,3}] = 0,$$

Can place in a **Memory Commutativity Matrix**

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Houshmand, Hosseini-Khayat, and Wilde. arXiv:1105.0649

More General Example (Ctd.)

Can reduce this matrix by full-rank binary matrices (row operations) to the following **normal form**:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Corresponds to the Pauli operators:

$$X_1,\ Z_1,\ X_2,\ Z_2,\ X_3,\ \text{and}\ Z_3$$

Do the row operations ***in reverse*** to get

$$\begin{aligned} g_{1,1} &= XIX, & g_{2,1} &= ZXZ, \\ g_{1,2} &= IIX, & g_{2,2} &= IIZ, \\ g_{1,3} &= IZI & g_{2,3} &= ZII \end{aligned}$$

Houshmand, Hosseini-Khayat, and Wilde. arXiv:1105.0649

More General Example (Ctd.)

Encoder finally requires just **three memory qubits**:

Mem.			Anc.		Info.	
I	I	I	Z	I	I	I
X	I	X	I	I	I	I
I	I	X	I	I	I	I
I	Z	I	I	I	I	I
I	I	I	I	Z	I	I
Z	X	Z	I	I	I	I
I	I	Z	I	I	I	I
Z	I	I	I	I	I	I

$\rightarrow$

Phys.				Mem.		
X	X	X	X	X	I	X
X	X	I	X	I	I	X
I	X	I	I	I	Z	I
I	I	X	X	I	I	I
Z	Z	Z	Z	Z	X	Z
Z	Z	I	Z	I	I	Z
I	Z	I	I	Z	I	I
I	I	Z	Z	I	I	I

The encoder is also non-catastrophic!

Follows from the previous argument:
All memory operators in a catastrophic cycle
should commute with the above operators and
are thus forced to be the identity operator.
(This same argument holds for full-rank memory commutativity matrices)

Houshmand, Hosseini-Khayat, and Wilde. arXiv:1105.0649

Summary

We have found a **simpler way to encode quantum convolutional codes**, and we have a general technique for finding a minimal-memory encoder for a given stabilizer representation of a quantum convolutional code.

In some cases, we can show that a minimal-memory encoder **always is non-catastrophic**.

Current Work

Would like to have a minimal memory encoder over all possible representations of the stabilizer (Houshmand has progress here)

Would like to have a general way for making a minimal-memory encoder be non-catastrophic as well (Some progress in arXiv:1105.0649)

Would like to have the same results for an online decoder corresponding to an online encoder