Strong converse exponents for a quantum channel discrimination problem

(joint w/ Cooney & Mosonyi in arXiv:1408.3373)

Umegaki ('62)
Relative entropy is a fundamental quantity in QIT, defined as

$$D(\rho\|\sigma) = \mathrm{Tr}\{\rho\,[\log\rho - \log\sigma]\}$$

Interesting that there are other generalizations of classical relative entropy such as Belavkin-Staszewski

$$D_{BS}(\rho\|\sigma) = \mathrm{Tr}\{\rho\,\log(\rho^{1/2}\sigma^{-1}\rho^{1/2})\}$$

and in fact many other ways to generalize...

What sets the Umegaki one apart is that it has an appealing operational interpretation in terms of the quantum Stein's lemma

the others don't...

Setting of quantum Stein's lemma is asymmetric quantum hypothesis testing.

What is hypothesis testing?

Someone prepares a quantum system in the state ρ or σ, but does not tell you which.

You then get to make a measurement $\{Q, I-Q\}$ on system.

$$\underset{\substack{\uparrow \\ \text{identify } \rho}}{Q} \qquad \underset{\substack{\uparrow \\ \text{identify } \sigma}}{I-Q}$$

There are two kinds of errors you can make:

(Type I) system prepared as ρ but measured as σ

$$\alpha(Q) \equiv \mathrm{Tr}\{(I-Q)\rho\}$$

(Type II) system prepared as σ but measured as ρ

$$\beta(Q) \equiv \mathrm{Tr}\{Q\sigma\}$$

There is a non-trivial tradeoff between the two kinds of errors

$$\alpha(Q) + \beta(Q) > 0 \quad \text{unless } \rho \perp \sigma$$

asymmetric setting explores the trade-off by imposing a constraint on Type I error & minimizing Type II error subject to the constraint.

Define

$$\beta_\varepsilon(\rho\|\sigma) = \min\{\beta(Q) : 0 \le Q \le I \wedge \alpha(Q) \le \varepsilon\}$$

can be useful to define the "hypothesis testing relative entropy" as

$$D_H^\varepsilon(\rho\|\sigma) = -\log \beta_\varepsilon(\rho\|\sigma)$$

obeys monotonicity under quantum operations

$$D_H^\varepsilon(\rho\|\sigma) \ge D_H^\varepsilon(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

b/c performing the optimal test can always do better than doing a channel & then a test quantum Stein's lemma applies to the case of distinguishing $\rho^{\otimes n}$ from $\sigma^{\otimes n}$ using a collective measurement $\{Q_n, I^{\otimes n} - Q_n\}$

Direct part of theorem: (Hiai-Petz '91)

$\exists$ a sequence of measurements $\{Q_n^*, \ldots\}$ such that for all $\varepsilon > 0$ & suff. large n

$$\alpha_n(Q_n) \leq \varepsilon$$

while $\beta_n(Q_n) \leq e^{-n[D(\rho\|\sigma) - \delta]}$ where $\delta > 0$

This gives the inequality

$$\lim_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n}) \geq D(\rho\|\sigma)$$

Strong converse part of theorem:

For any sequence of measurements $\{Q_n, \ldots\}$ such that

$$\beta_n(Q_n) \leq e^{-n[D(\rho\|\sigma)+\delta]}$$

w/ $\delta > 0$

then $\alpha_n(Q_n) \rightarrow 1$

so that there is no trade-off

equivalent statement of strong converse is

$$\lim_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq D(\rho\|\sigma)$$

can prove it somewhat easily via the Renyi relative entropy & the following lemma

$$D_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha-1}\log \mathrm{Tr}\{\rho^\alpha \sigma^{1-\alpha}\}$$

$$D_H^{\varepsilon}(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log\left(\frac{1}{1-\varepsilon}\right)$$

for $\alpha > 1$ & $\varepsilon \in (0,1)$

How to prove: Choose $\{Q, I-Q\}$ to be optimum for D_H^{ε}

Apply to tensor-power states & regularize:

$$\frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \leq \frac{1}{n} D_\alpha(\rho^{\otimes n} \| \sigma^{\otimes n}) + \frac{1}{n}\left(\frac{\alpha}{\alpha-1}\right) \log\left(\frac{1}{1-\varepsilon}\right)$$

$$= D_\alpha(\rho\|\sigma) + \frac{1}{n}(\ \cdots\ )$$

Take limit as $n \to \infty$ to get

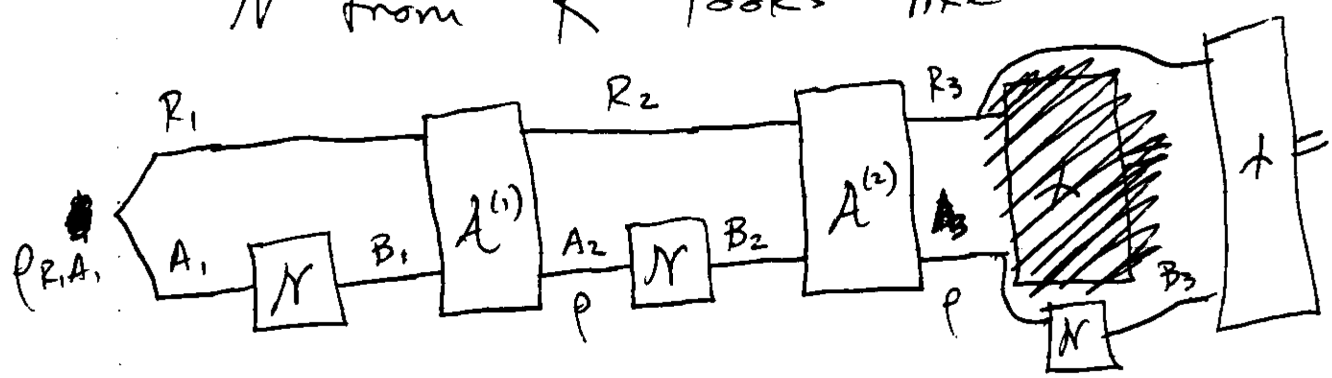
$$\lim_{n\to\infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n} \| \sigma^{\otimes n}) \leq D_\alpha(\rho\|\sigma)$$

but holds for every α, so take limit as $\alpha \searrow 1$ to get result since

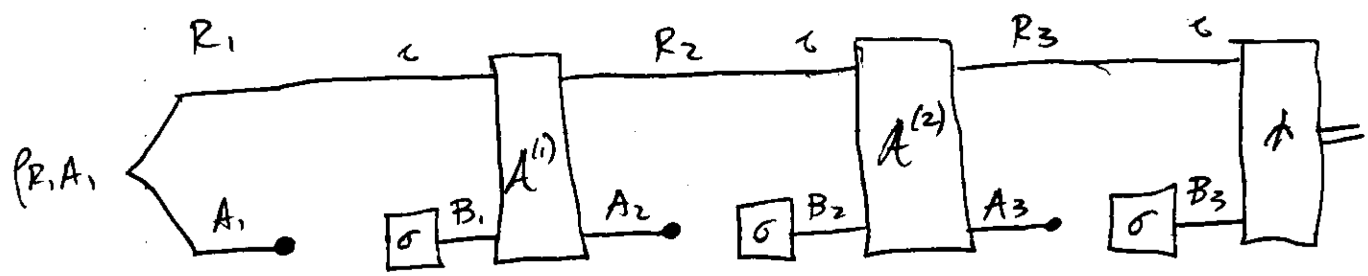
$$\lim_{\alpha \searrow 1} D_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$$

New Setting we are interested in is adaptive channel discrimination between channel $\mathcal{N}$ + replacer channel $\mathcal{R}(X) \equiv \mathrm{Tr}\{X\} \cdot \sigma$

General strategy for discriminating $\mathcal{N}$ from $\mathcal{R}$ looks like



vs.



We would like to establish a Stein's lemma in this setting

Direct part is easy:

Just use the old Stein's lemma

w/ $\phi_{RA}^{\otimes n}$ & no adaptation

if channel is $\mathcal{N}$, then output is $\left[\mathcal{N}_{A\to B}(\phi_{RA})\right]^{\otimes n}$

if channel is $\mathcal{R}$, then output is $\left[\phi_R \otimes \sigma_B\right]^{\otimes n}$

& optimal decay exponent is

$$\max_{\phi_{RA}} D\left(\mathcal{N}_{A\to B}(\phi_{RA}) \,\|\, \phi_R \otimes \sigma_B\right)$$

↑ can optimize...

Strong converse part:

use "sandwiched" Renyi relative entropy

from arXiv: 1306.3142 &
arXiv: 1306.1586

$$\widetilde{D}_\alpha(\rho\|\sigma) \equiv \frac{1}{\alpha-1}\log \operatorname{Tr}\left\{\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right\}$$

$$= \frac{\alpha}{\alpha-1}\log\left\|\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right\|_\alpha$$

w/ $\|A\|_\alpha \equiv \operatorname{Tr}\{|A|^\alpha\}^{1/\alpha}$

use lemma

$$D_H^{\varepsilon}(\rho\|\sigma) \leq \widetilde{D}_\alpha(\rho\|\sigma) + \left(\frac{\alpha}{\alpha-1}\right)\log\left(\frac{1}{1-\varepsilon}\right)$$

Apply this lemma to the last step of picture

$$D_H^{\varepsilon}\left(\mathcal{N}_{A_3\to B_3}(\rho_{R_3A_3}) \,\|\, \tau_{R_3}\otimes\sigma_{B_3}\right)$$

$$\leq \widetilde{D}_\alpha\left(\mathcal{N}_{A_3\to B_3}(\rho_{R_3A_3}) \,\|\, \tau_{R_3}\otimes\sigma_{B_3}\right) \quad (*)$$

$$+\left(\frac{\alpha}{\alpha-1}\right)\log\left(\frac{1}{1-\varepsilon}\right)$$

focus on & rewrite as

$$\frac{\alpha}{\alpha-1}\log\left\|(\tau_{R_3}\otimes\sigma_{B_3})^{\frac{1-\alpha}{2\alpha}}\,\mathcal{N}_{A_3\to B_3}(\rho_{R_3A_3})\,(\tau_{R_3}\otimes\sigma_{B_3})^{\frac{1-\alpha}{2\alpha}}\right\|_\alpha$$

$$= \frac{\alpha}{\alpha-1}\log\left\|\sigma_{B_3}^{\frac{1-\alpha}{2\alpha}}\,\mathcal{N}_{A_3\to B_3}\left(\tau_{R_3}^{\frac{1-\alpha}{2\alpha}}\rho_{R_3A_3}\tau_{R_3}^{\frac{1-\alpha}{2\alpha}}\right)\sigma_{B_3}^{\frac{1-\alpha}{2\alpha}}\right\|_\alpha$$

$$= \frac{\alpha}{\alpha-1}\log\left\|\left(\Theta_\sigma\circ\mathcal{N}_{A_3\to B_3}\right)\left(\tau_{R_3}^{\frac{1-\alpha}{2\alpha}}\rho_{R_3A_3}\tau_{R_3}^{\frac{1-\alpha}{2\alpha}}\right)\right\|_\alpha$$

where $\Theta_\sigma(X) \equiv \sigma_{B_3}^{\frac{1-\alpha}{2\alpha}}(X)\,\sigma_{B_3}^{\frac{1-\alpha}{2\alpha}}$

Focus on expression in log &
rewrite as

$$\frac{\left\| \left(\Theta_\sigma \circ \mathcal{N}_{A_3 \to B_3}\right)\left(\tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \rho_{R_3 A_3} \tau_{R_3}^{\frac{1-\alpha}{2\alpha}}\right)\right\|_\alpha}{\left\| \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \rho_{R_3} \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha} \cdot \left\| \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \rho_{R_3} \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha$$

$$\leq \left[\sup_{X_{R_3 A_3} > 0} \frac{\left\| \left(\Theta_\sigma \circ \mathcal{N}_{A_3 \to B_3}\right)\left(X_{R_3 A_3}\right)\right\|_\alpha}{\left\| X_{R_3} \right\|_\alpha}\right] \cdot \left\| \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \rho_{R_3} \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha$$

$$= \left\| \Theta_\sigma \circ \mathcal{N} \right\|_{CB, 1 \to \alpha} \cdot \left\| \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \rho_{R_3} \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha$$

plug back into log to find that (*)

$$\leq \frac{\alpha}{\alpha-1} \log \left\| \Theta_\sigma \circ \mathcal{N} \right\|_{CB, 1\to\alpha} + \frac{\alpha}{\alpha-1} \log \left\| \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \rho_{R_3} \tau_{R_3}^{\frac{1-\alpha}{2\alpha}} \right\|_\alpha$$

$$= \quad '' \quad + \widetilde{D}_\alpha\left(\rho_{R_3} \| \tau_{R_3}\right)$$

$$\leq \quad '' \quad + \widetilde{D}_\alpha\left(\mathcal{N}_{A_2 \to B_2}\left(\rho_{R_2 A_2}\right) \| \tau_{R_2} \otimes \sigma_{B_2}\right)$$

monotonicity of $\widetilde{D}_\alpha$ under $\mathrm{Tr}_{A_3} \circ \mathcal{A}^{(2)}$ $\forall \alpha > 1$

10a

CB, $1 \to \alpha$ norm defined as
of $\mathcal{M}$

$$\| \mathcal{M} \|_{CB, 1 \to \alpha} = \max_{\rho_R} \left\| \left(\rho_R^{1/2\alpha} \otimes I_B \right) \Gamma^{\mathcal{M}}_{RB} \left(\rho_R^{1/2\alpha} \otimes I_B \right) \right\|_\alpha$$

where $\Gamma^{\mathcal{M}}_{RB} \equiv \mathcal{M}_{A \to B} \left(\Gamma_{RA} \right)$

& $|\Gamma\rangle_{RA} \equiv \sum_i |i\rangle_R |i\rangle_A$

Iterate this "n times" to get

$$D_H^{\varepsilon}\left(\mathcal{N}_{A_n \to B_n}(\rho_{R_n A_n}) \,\|\, \tau_{R_n} \otimes \sigma_{B_n} \right)$$

$$\leq n \cdot \frac{\alpha}{\alpha - 1} \log \| \Theta_\sigma \circ \mathcal{N} \|_{CB, 1 \to \alpha} + \left(\frac{\alpha}{\alpha - 1} \right) \log \left(\frac{1}{1 - \varepsilon} \right)$$

$$= n \cdot \sup_{\phi_{RA}} \widetilde{D}_\alpha \left(\mathcal{N}_{A \to B}(\phi_{RA}) \,\|\, \phi_R \otimes \sigma_B \right)$$

Divide by n & take limit as $n \to \infty$ & then as $\alpha \to 1$ to get upper bound of

$$\sup_{\phi_{RA}} D\left(\mathcal{N}_{A \to B}(\phi_{RA}) \,\|\, \phi_R \otimes \sigma_B \right)$$

---

applications are in quantum illumination to demonstrate that tensor-power strategies are optimal

can also show strong converse for quantum feedback assisted communication