

Sequential, successive, and simultaneous  
decoders for entanglement-assisted classical  
communication  
(arXiv:1107.1347)

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# Packing Argument for Sequential Decoding

A generalization of Giovannetti *et al.* <sup>1</sup>

$$\rho_m \xrightarrow{\Pi\Pi_1?} \rho_m \xrightarrow{\Pi\Pi_2?} \dots \xrightarrow{\Pi\Pi_{m-1}^?} \rho_m \xrightarrow{\Pi\Pi_m^?} \text{YES!}$$

Given an ensemble  $\{p_X(x), \rho_x\}_{x \in \mathcal{X}}$  where  $\rho \equiv \sum_{x \in \mathcal{X}} p_X(x) \rho_x$ . If there exists projectors  $\Pi$  and  $\{\Pi_x\}_{x \in \mathcal{X}}$  such that:

$$\begin{aligned} \text{Tr}\{\Pi\rho_x\} &\geq 1 - \epsilon, & \Pi_x\rho_x\Pi_x &\geq \frac{1}{d}\Pi_x, \\ \text{Tr}\{\Pi_x\rho_x\} &\geq 1 - \epsilon, & \Pi\rho\Pi &\leq \frac{1}{D}\Pi, \\ & & [\Pi_x, \rho_x] &= 0, \end{aligned}$$

then we can encode messages from a set  $\mathcal{M}$  with  $|\mathcal{M}| \ll D/d$  using this ensemble and sequential decoding gives us that:

$$\mathbb{E}\{p_{\text{succ}}\} \geq \left| (1 - 2\epsilon) \left( 2 - e^{\frac{d}{D}|\mathcal{M}|} \right) \right|^2$$

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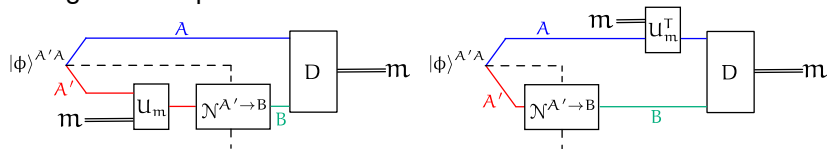
<sup>1</sup>Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Achieving the Holevo bound via sequential measurements. December 2010. arXiv:1012.0386.

# Sequential Decoding for EA Classical Communication

Many copies of the shared entanglement. For each message  $m \in \mathcal{M}$ , we randomly choose an encoder as in Hsieh *et al.*<sup>2</sup>:

$$|\phi\rangle^{A'A} = \sum_t \sqrt{p(t)} |\Phi_t\rangle^{A'A}, \quad U_m \equiv \bigoplus_t (-1)^{b_t} X(x_t) Z(z_t).$$

Using the transpose trick:



Decoding projectors:

$$\Pi \equiv \Pi_\delta^A \otimes \Pi_\delta^B, \quad \Pi_m \equiv U_m^\dagger \Pi_\delta^{AB} U_m^*.$$

<sup>2</sup>Min-Hsiu Hsieh, Igor Devetak, and Andreas Winter. Entanglement-assisted capacity of quantum multiple-access channels. *IEEE Transactions on Information Theory*, 54(7):3078-3090, 2008.

# Sequential Decoding for EA Classical Communication over a MAC

$$\rho_{l,m} \xrightarrow{\Pi \Pi_1?} \rho_{l,m} \xrightarrow{\Pi \Pi_2?} \dots \xrightarrow{\Pi \Pi_{l-1}^?} \rho_{l,m} \xrightarrow{\Pi \Pi_l^?} \text{YES!}$$

$$\rho_{l,m} \xrightarrow{\Pi_l \Pi_{l,1}^?} \rho_{l,m} \xrightarrow{\Pi_l \Pi_{l,2}^?} \dots \xrightarrow{\Pi_l \Pi_{l,m-1}^?} \rho_{l,m} \xrightarrow{\Pi_l \Pi_{l,m}^?} \text{YES!}$$

Given is  $\{p_X(x)p_Y(y), \rho_{x,y}\}$ , with some new requirements for the MAC setting:

$$\Pi_x \rho_x \Pi_x \geq \frac{1}{d_1^-} \Pi_x,$$

$$\Pi_{x,y} \rho_{x,y} \Pi_{x,y} \geq \frac{1}{d_2} \Pi_{x,y},$$

$$\Pi \rho \Pi \leq \frac{1}{D_1} \Pi,$$

$$\Pi_x \rho_x \Pi_x \leq \frac{1}{d_1^+} \Pi_{x,y},$$

$$[\Pi_x, \rho_x] = 0,$$

$$[\Pi_{x,y} \rho_{x,y}] = 0.$$

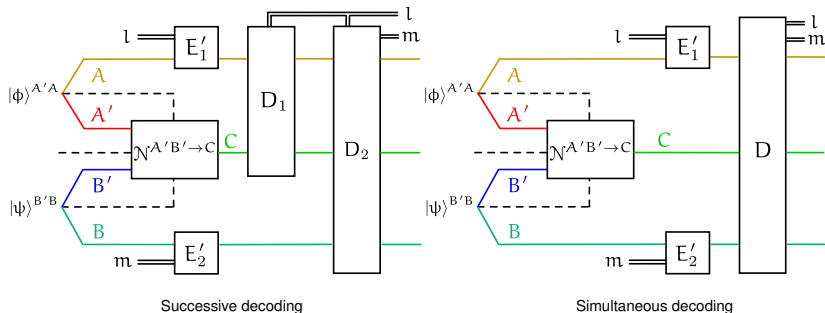
We then can encode message sets with size  $|\mathcal{L}| \ll D_1/d_1^-$  and  $|\mathcal{M}| \ll d_1^+/d_2$ .

# Simultaneous Decoding for EA Classical Communication

Two message sets  $\mathcal{L}$  and  $\mathcal{M}$ . Shared entanglement sent through a noisy MAC. Same encoding unitary as above with the transpose trick:

$$\rho^{ABC} \equiv \mathcal{N}^{A'B' \rightarrow C} \left( |\phi\rangle\langle\phi|^{A'A} \otimes |\psi\rangle\langle\psi|^{B'B} \right),$$

$$\sigma_{lm} \equiv (U_l^T \otimes U_m^T) \rho^{ABC} (U_l^* \otimes U_m^*).$$



# Simultaneous Decoding for EA Classical Communication

This time we employ a *simultaneous* decoder:

$$\Lambda_{lm} \equiv \left( \sum_{l',m'} \gamma_{l'm'} \right)^{-\frac{1}{2}} \gamma_{lm} \left( \sum_{l',m'} \gamma_{l'm'} \right)^{-\frac{1}{2}},$$

$$\gamma \equiv u_l^T \hat{\Pi}_3 \hat{\Pi}_2 u_m^T \hat{\Pi} u_m^* \hat{\Pi}_2 \hat{\Pi}_3 u_l^*,$$

$$\hat{\Pi}_1 \equiv \Pi^A \otimes \Pi^{BC}, \quad \hat{\Pi}_2 \equiv \Pi^B \otimes \Pi^{AC}, \quad \hat{\Pi}_3 \equiv \Pi^C \otimes \Pi^{AB},$$

$$\hat{\Pi} \equiv \Pi^{ABC},$$

with the average error probability:  $\frac{1}{|\mathcal{L}| \cdot |\mathcal{M}|} \sum_{l,m} \text{Tr}\{(I - \Lambda_{lm}) \sigma_{lm}\}$ ,

# Simultaneous Decoding for EA Classical Communication

Instead of analyzing  $\sigma_{lm}$  directly, we “smooth”  $\sigma_{lm}$  into

$$\begin{aligned}\theta_{lm} &\equiv U_m^T \hat{\Pi}_1 U_m^* \sigma_{lm} U_m^* \hat{\Pi}_1 U_m^T \\ &= U_m^T \hat{\Pi}_1 U_l^T \rho^{ABC} U_l^* \hat{\Pi}_1 U_m^*,\end{aligned}$$

and use the following inequality

$$\mathrm{Tr}\{\Gamma\rho\} \leq \mathrm{Tr}\{\Gamma\sigma\} + \|\rho - \sigma\|_1.$$

Using the Hayashi-Nagaoka inequality:

$$\bar{p}_e \leq \frac{1}{|\mathcal{L}| \cdot |\mathcal{M}|} \sum_{l,m} \left( 2\mathrm{Tr}\{(I - \Upsilon_{lm})\theta_{lm}\} + 4 \sum_{\substack{(l',m') \\ \neq (l,m)}} \mathrm{Tr}\{\Upsilon_{l'm'}\theta_{lm}\} \right) + \epsilon.$$

# Simultaneous Decoding in Quantum Communication

We can then run our protocol coherently à la Harrow<sup>3</sup>

$$\langle \mathcal{N} \rangle + H(A) [qq]_{AC} + H(B) [qq]_{BC} \geq R_1 [q \rightarrow qq]_{AC} + R_2 [q \rightarrow qq]_{BC}.$$

We then apply the coherent communication identity,

$$2 [q \rightarrow qq] = [q \rightarrow q] + [qq],$$

to obtain catalytic protocols for assisted and unassisted quantum communication.

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<sup>3</sup>A. W. Harrow. Coherent communication of classical messages. Physical Review Letters, 92:097902, 2004.



## EA Bosonic MAC

We give an achievable rate region for EA classical communication over a Bosonic MAC. We consider a beam-splitter

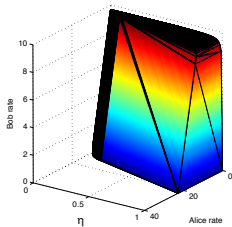
$$\begin{aligned}\hat{c} &= \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{b}, \\ \hat{e} &= -\sqrt{1-\eta}\hat{a} + \sqrt{\eta}\hat{b},\end{aligned}$$

with a two-mode squeezed vacuum as the input state:

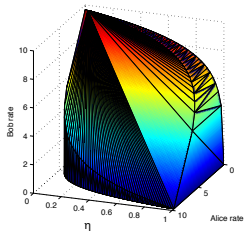
$$\sum_{n=0}^{\infty} \sqrt{\frac{N_S^n}{(N_S + 1)^{n+1}}} |n\rangle |n\rangle.$$

Each sender shares a two-mode squeezed state with the receiver.

# EA Bosonic MAC

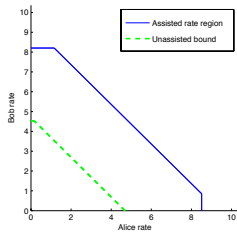


(a)

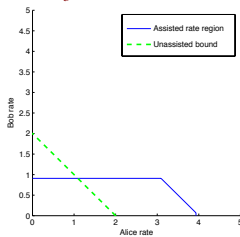


(b)

Achievable rate region for  $\eta \in [0, 1]$ : (a)  $N_{S_a} = 1000, N_{S_b} = 10$ . (b)  $N_{S_a} = 10, N_{S_b} = 10$ .



(a)



(b)

Comparison with Yen-Shapiro outer bound: (a)  $N_{S_a} = 10, N_{S_b} = 8, \eta = 1/2$ . (b)  $N_{S_a} = 1, N_{S_b} = 1, \eta = 0.95$ .

# Conclusions

In conclusion, we gave

- ▶ Extension to the sequential decoding and its applications.
- ▶ An EA simultaneous decoder and its applications.
- ▶ An achievable rate region for EA classical communication over a bosonic MAC and comparison with the unassisted case.

We should also mention that Sen <sup>4</sup> has recently improved the analysis for sequential decoding, and independently discovered an unassisted simultaneous decoder using sequential decoding.

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<sup>4</sup>Pranab Sen. Achieving the Han-Kobayashi inner bound for the quantum interference channel by sequential decoding. September 2011. arXiv:1109.0802.