Information Trade-offs for Optical Quantum Communication

Mark M. Wilde

School of Computer Science
McGill University

In collaboration with
Patrick Hayden and Saikat Guha

arXiv:1105.0119

Biannual INTRIQ meeting,
McGill University, October 3, 2011
Overview

Review of basic protocols

Review of classical capacity, quantum capacity, and entanglement-assisted capacity of bosonic channels

The idea of encoding both classical and quantum data, etc.

Bosonic channels have good trade-off curves
Entanglement Distribution

Trivial way to generate entanglement between Alice and Bob

\[ |\Phi\rangle^{AB} \equiv \frac{1}{\sqrt{2}} \left( |00\rangle^{AB} + |11\rangle^{AB} \right) \]

(an ebit)
Super-dense Coding

One noiseless ebit and one noiseless qubit channel generates two classical bit channels

Bennett and Wiesner, Physical Review Letters 69, 2881 (1992)
Teleportation

One noiseless ebit and two classical bit channels generates a noiseless qubit channel from Alice to Bob

Bennett et al., Physical Review Letters 70, 1895 (1993)
Noisy Quantum Channel Model

Model channel as a completely positive, trace-preserving map

Alice inputs a density operator: \( \rho_x^A \)

Bob gets density operator: \( \rho_x^B \)

How much information can Alice transmit to Bob (capacity)?
What is Capacity?

The ultimate rate at which two parties can communicate or perform some given task (optimized over all possible encodings and decodings).

Capacity theorem has two parts:

Direct Coding Theorem – For any rate below capacity, there exists a coding scheme that achieves that rate with vanishing error.

Converse Theorem – For any coding scheme with vanishing error, its rate is below capacity.
Sending Classical Information over a Quantum Channel

Coding Strategy
(similar to that for classical case)

Use the channel many times so that law of large numbers comes into play

Code randomly with an ensemble of the following form:

\[ \{ p(x), \rho_x^{A'} \} \quad x \in \chi \]

Channel input states are **product states**

Allow for small error but show that the error vanishes with large block length

_Schumacher & Westmoreland, PRA, 56, 131-138 (1997)._
Encoder just maps classical signal to a tensor product state

Decoder performs a measurement over all the output states to determine transmitted classical signal
Can achieve the following rate (bits/channel use):

\[ I(X;B) \sigma \]

where \[ \sigma^{XB} = \sum_{x \in \mathcal{X}} p(x) |x\rangle \langle x|^{X} \otimes \mathcal{N}^{A' \rightarrow B}(\rho_{x}') \]

Holevo information of the channel:

\[ \chi(\mathcal{N}) \equiv \max_{\sigma} I(X;B) \sigma \]

Capacity of the channel with product input states:

\[ C(\mathcal{N}) = \chi(\mathcal{N}) \]

Capacity of the channel with entangled input states:

\[ C(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} \chi(\mathcal{N}^{\otimes n}) \]
Bosonic Channels

Lossy Bosonic Channel  
(models fiber optic or free space transmission)

Thermalizing Channel  
(similar model with background radiation)

Amplifier Channel  
(models amplifier noise, Hawking-Unruh radiation)

Weedbrook et al., Gaussian Quantum Information, Reviews of Modern Physics (2011).
Sending Classical Data over Bosonic Channels

Classical capacity of **lossy bosonic channel** is exactly

\[ g(\eta N_S) \]

where \( \eta \) is **transmissivity** of channel, \( N_S \) is the **mean input photon number**, and \( g(x) = (x+1) \log(x+1) - x \log x \) is the **entropy** of a **thermal state** with photon number \( x \).

Can achieve this capacity by selecting **coherent states** randomly according to a complex, isotropic Gaussian prior with variance \( N_S \).

Preserving entanglement is the same as transmitting quantum data

$$\mathcal{N}^{A'\rightarrow B}(\phi AA')$$

**Coherent information** of a quantum channel:

$$Q(\mathcal{N}) \equiv \max_{\phi} I(A\rangle B)$$

where

$$I(A\rangle B) \equiv H(B) - H(AB)$$

Sending Quantum Data over Bosonic Channels

Quantum capacity of lossy bosonic channel is

\[ g(\eta N_S) - g((1 - \eta) N_S) \]

**Interpretation:** Generate random quantum codes from a thermal state distribution

An achievable rate is the difference of Bob and Eve's entropy

Sending Quantum Data with Entanglement Assistance

Encoder is a random unitary mapping

Decoder decouples from Eve the quantum information Alice would like to protect

Father Protocol

Can achieve the following resource inequality:

\[ \langle N^{A' \rightarrow B} \rangle + \frac{1}{2} I(A; E)_\psi[qq] \geq \frac{1}{2} I(A; B)_\psi[q \rightarrow q] \]

where

Entanglement-Assisted Quantum Transmission over Bosonic Channels

Entanglement-Assisted Quantum Capacity:

\[
\frac{1}{2} \left[ g(\eta N_S) + g(N_S) - g((1 - \eta) N_S) \right]
\]

Again generate random quantum codes from a thermal distribution

Prior shared entanglement boosts capacity

Wilde, Hayden, Guha. arXiv:1105.0119
The CQE Trade-off Setting

\[ nC = \log |M| - \log |L| \]
\[ nQ = \log |A_1| - \log |A_2| \]
\[ nE = \log |S_A| - \log |T_A| \]

Quantum Dynamic Capacity Theorem

The dynamic capacity region $C_{\text{CQE}}(\mathcal{N})$ is

$$C_{\text{CQE}}(\mathcal{N}) = \bigcup_{k=1}^{\infty} \frac{1}{k} C_{\text{CQE}}^{(1)}(\mathcal{N} \otimes^k).$$  \hspace{1cm} (1)

The “one-shot” region $C_{\text{CQE}}^{(1)}(\mathcal{N})$ is

$$C_{\text{CQE}}^{(1)}(\mathcal{N}) = \bigcup_{\sigma} C_{\text{CQE}, \sigma}^{(1)}(\mathcal{N}).$$

The “one-shot, one-state” region $C_{\text{CQE}, \sigma}^{(1)}(\mathcal{N})$ is the set of all rates $C$, $Q$, and $E$, such that

$$C + 2Q \leq I(AX; B)_\sigma,$$  \hspace{1cm} (2)

$$Q + E \leq I(AB; X)_\sigma,$$  \hspace{1cm} (3)

$$C + Q + E \leq I(X; B)_\sigma + I(A; BX)_\sigma.$$  \hspace{1cm} (4)

The above entropic quantities are with respect to a classical-quantum state $\sigma^{XAB}$ where

$$\sigma^{XAB} = \sum_x p(x) |x\rangle \langle x|^X \otimes \mathcal{N}^{A' \rightarrow B}(\phi_{x}^{AA'}).$$  \hspace{1cm} (5)

One should consider states on $A'^k$ instead of $A'$ when taking the regularization.
Achievability

There exists a protocol for entanglement-assisted classical and quantum communication that achieves the following rates:

\[ \langle N^{A' \rightarrow B} \rangle + \frac{1}{2} I(A; E|X)_{\sigma}[qq] \geq \frac{1}{2} I(A; B|X)_{\sigma}[q \rightarrow q] + I(X; B)_{\sigma}[c \rightarrow c] \]

Combine this with teleportation, dense coding, and entanglement distribution...

Trade-off Coding for Dephasing Channels

Classical-Quantum Trade-off

Trade-offs for a qubit dephasing channel with various noise levels just barely beat time-sharing

Why then would you implement a trade-off coding strategy in practice?

Trade-off Coding for Bosonic Channels

Classical-quantum trade-off for the lossy bosonic channel with $\eta = 3/4$

Classical-entanglement trade-off for the lossy bosonic channel with $\eta = 3/4$

Trade-off is so strong for bosonic channels that it would be silly not to use such a strategy

Wilde, Hayden, Guha. arXiv:1105.0119
Power-Sharing Coding Strategy

Coding Ensemble:

\[
\left\{ p_{(1-\lambda)N_S}(\alpha), D^{A'}(\alpha) \left| \psi_{\text{TMS}} \rightangle^{AA'} \right\}.
\]

where

\[
p_{(1-\lambda)N_S}(\alpha) \equiv \frac{1}{\pi(1-\lambda)N_S} \exp \left\{ -|\alpha|^2 / (1-\lambda)N_S \right\}
\]

\[
\left| \psi_{\text{TMS}} \rightangle^{AA'} \equiv \sum_{n=0}^{\infty} \sqrt{\frac{[\lambda N_S]^n}{[\lambda N_S + 1]^{n+1}}} \left| n \rightangle^A \left| n \rightangle^{A'}
\]

Wilde, Hayden, Guha. arXiv:1105.0119
Achievable Rate Region for Lossy Channel

\[ C + 2Q \leq g(\lambda N_S) + g(\eta N_S) - g((1 - \eta) \lambda N_S), \]
\[ Q + E \leq g(\eta \lambda N_S) - g((1 - \eta) \lambda N_S), \]
\[ C + Q + E \leq g(\eta N_S) - g((1 - \eta) \lambda N_S) \]

\( \lambda \) is a power-sharing parameter between zero and one

Wilde, Hayden, Guha. arXiv:1105.0119
Rule of Thumb for Trade-off Coding

To be within $\epsilon$ bits of quantum capacity, choose

$$\lambda = 1 / [\eta (1 - \eta) \epsilon N_S \ln 2]$$

To be within $\epsilon$ bits of EA capacity, choose

$$\lambda = 5 / [6 \epsilon N_S (1 - \eta) \ln 2]$$

Classical–quantum trade–off for the lossy bosonic channel

Classical–entanglement trade–off for the lossy bosonic channel

Wilde, Hayden, Guha. arXiv:1105.0119
Conclusion

**Power-sharing** significantly outperforms **time-sharing** between the best known protocols.

Do there exist structured encoders and decoders to achieve these rates?

Where to learn more about **quantum Shannon theory**:
Book available at arXiv:1106.1445