

Quantum interference channels

Omar Fawzi*, Patrick Hayden*, Ivan Savov*, Pranab Sen*[†] and Mark M. Wilde*

* School of Computer Science, McGill University, *Montréal, Québec, Canada*

[†] School of Technology and Computer Science, Tata Institute of Fundamental Research, *Mumbai, India*

Abstract—The discrete memoryless interference channel is modelled as a conditional probability distribution with two outputs depending on two inputs and has widespread applications in practical communication scenarios. In this paper, we introduce and study the quantum interference channel, a generalization of a two-input, two-output memoryless channel to the setting of quantum Shannon theory. We discuss three different coding strategies and obtain corresponding achievable rate regions for quantum interference channels. We calculate the capacity regions in the special cases of “very strong” and “strong” interference. The achievability proof in the case of “strong” interference exploits a novel quantum simultaneous decoder for two-sender quantum multiple access channels. We formulate a conjecture regarding the existence of a quantum simultaneous decoder in the three-sender case and use it to state the rates achievable by a quantum Han-Kobayashi strategy.

I. INTRODUCTION

Modern communication systems usually approach the problem of inter-carrier interference by treating the interfering signals as noise. Indeed, techniques like code division multiple access aim to make the encoded signals as similar to background noise as possible by spreading the signal power over large sections of the spectrum. Rather than treating the interference as noise, a receiver could instead try to decode the interfering signals and then “subtract” them from the received signal in order to reduce (or even remove) the interference. The development of these ideas into practical codes for M -user interference channels would have profound implications for many areas of communications engineering.

The theory of this problem has been studied for more than 30 years, in particular for channels with two senders and two receivers [1], [2]. The approach of completely decoding the interfering messages applies to channels with “very strong” interference, and it is optimal for this class of channels [3]. For an arbitrary interference channel, it may only be possible to *partially* decode the interfering signal. Still, the receivers can achieve better communication rates using this side information when decoding the messages intended for them. The best achievable rate region for the general interference channel is based on partial decoding of the interference and is due to Han and Kobayashi [4].

In this paper, we apply and extend some insights from classical information theory to the study of quantum interference channels (QICs). These channels can model physical systems such as fibre-optic cables and free space optical communication channels, when operating in low-power regimes [5]. Inspired by results like the Holevo-Schumacher-

Westmoreland theorem on the classical capacity of point-to-point channels [6], [7], and Winter’s results on the capacity of quantum multiple access channels [8], we propose the study of classical communication over quantum interference channels.

We structure this paper as follows. In Section II we review our main results. Section III introduces notation and defines the key concepts. In Section IV we discuss the quantum multiple access channel, and the difference between successive decoding, simultaneous decoding and rate-splitting approaches to achieving the capacity. Section V presents our results on the quantum interference channel. We conclude by stating open problems in Section VI.

II. SUMMARY OF RESULTS

We initiate the study of quantum interference channels, a fundamental problem of multiuser communication theory. As first steps in this study, we prove the capacity region for channels with “very strong” interference (Theorem 4) and channels with “strong” interference (Theorem 6). For general interference channels we obtain a quantum analogue of Sato’s outer bound (Theorem 5) and an achievable rate region inspired by the Han-Kobayashi coding strategy [4] and rate-splitting [9]. Our work serves to highlight the importance of quantum simultaneous decoding for the multiple access channel as a key ingredient for the construction of the interference channel codes. Prior results on quantum multiple access channels are based on successive decoding and time-sharing [8], but in Theorem 2 we show that a quantum simultaneous decoder exists for multiple access channels with two senders. The quantum Han-Kobayashi coding strategy (Theorem 7) requires the use of a quantum simultaneous decoder for multiple access channels with three senders. It is not obvious how to extend the techniques used to prove Theorem 2 to the three-sender case. We formulate Conjecture 3 concerning the existence of a quantum simultaneous decoder for three-sender quantum multiple access channels. A proof of this conjecture would have profound consequences for multiuser quantum information theory since it would allow for many classical information theory results based on simultaneous decoding to be adapted to the quantum setting.

III. PRELIMINARIES

In this section, we define the quantum interference channel and the communication task that we are trying to achieve.

1) *Notation:* We denote quantum systems as A , B , and C and their corresponding Hilbert spaces as \mathcal{H}^A , \mathcal{H}^B , and \mathcal{H}^C . We represent quantum states of the system A as a density operator ρ^A , which is a positive semi-definite operator with unit trace. We model our lack of access to a quantum system with the partial trace operation. Given a state ρ^{AB} shared between Alice and Bob, we can describe Alice's state with the reduced density operator $\rho^A = \text{Tr}_B \{\rho^{AB}\}$, where Tr_B denotes a partial trace over Bob's degrees of freedom. Let $H(A)_\rho \equiv -\text{Tr} \{\rho^A \log \rho^A\}$ denote the von Neumann entropy of the state ρ^A . A noiseless quantum operation is represented by a unitary operator U which acts on a state ρ by conjugation $U\rho U^\dagger$, which we denote as $U \cdot \rho \equiv U\rho U^\dagger$. Noisy quantum operations are represented by completely positive trace-preserving (CPTP) maps $\mathcal{N}^{A' \rightarrow B}$, which accept input states in A' and produce output states in B . Let $\text{conv}(\mathcal{R})$ denote the convex closure of any geometrical region \mathcal{R} . Throughout this paper, logarithms and exponents are taken base two unless otherwise specified.

2) *Definitions:* The classical discrete memoryless interference channel (IC) is described by a triple $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1, y_2 | x_1, x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$, where \mathcal{X}_i is a finite set of possible input symbols for Sender i and \mathcal{Y}_j is the set of possible output symbols for Receiver j .

If we extend this definition to allow both inputs and outputs to be quantum systems we obtain the following:

Definition 1. A two party quantum interference channel is a triple $(\mathcal{H}^{A'_1} \otimes \mathcal{H}^{A'_2}, \mathcal{N}^{A'_1 A'_2 \rightarrow B_1 B_2}, \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2})$, where A'_1 and A'_2 are the two quantum systems that are input to the channel by the senders, B_1 and B_2 are the output systems, and $\mathcal{N}^{A'_1 A'_2 \rightarrow B_1 B_2}$ is a completely positive trace-preserving (CPTP) map.

A simpler channel is the *classical-quantum (c-q) interference channel*, where only the outputs are quantum.

Definition 2. A two party cc-qq interference channel is a triple $(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{N}^{X_1 X_2 \rightarrow B_1 B_2}(x_1, x_2) \equiv \rho_{x_1, x_2}^{B_1 B_2}, \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2})$, which models a general communication network with two classical inputs and a quantum state $\rho_{x_1, x_2}^{B_1 B_2}$ as output.

In this paper, we focus our attention on the class of classical-quantum interference channels, though generalizations of our results to channels with quantum inputs are straightforward. We fully specify a *cc-qq* interference channel by the set of output states it produces $\{\rho_{x_1, x_2}^{B_1 B_2}\}_{x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2}$. A classical interference channel with transition probability $p(y_1, y_2 | x_1, x_2)$ is a special case of the *cc-qq* channel where the output states are of the form $\rho_{x_1, x_2}^{B_1 B_2} = \sum_{y_1, y_2} p(y_1, y_2 | x_1, x_2) |y_1\rangle\langle y_1|^{B_1} \otimes |y_2\rangle\langle y_2|^{B_2}$ where $\{|y_1\rangle\}$ and $\{|y_2\rangle\}$ are orthonormal bases of \mathcal{H}^{B_1} and \mathcal{H}^{B_2} .

3) *Information processing task:* The task of communication over an interference channel can be described as follows. Using n independent uses of the channel, the objective is for Sender 1 to communicate with Receiver 1 at a rate R_1 and for Sender 2 to communicate with Receiver 2 at a rate R_2 . More specifically, Sender 1 chooses a message m_1 from a message set $\mathcal{M}_1 \equiv \{1, 2, \dots, |\mathcal{M}_1|\}$ where $|\mathcal{M}_1| = 2^{nR_1}$,

and Sender 2 similarly chooses a message m_2 from a message set $\mathcal{M}_2 \equiv \{1, 2, \dots, |\mathcal{M}_2|\}$ where $|\mathcal{M}_2| = 2^{nR_2}$. Senders 1 and 2 encode their messages as codewords $x_1^n(m_1) \in \mathcal{X}_1^n$ and $x_2^n(m_2) \in \mathcal{X}_2^n$ respectively, which are then input to the channel. The output of the channel is an n -fold tensor product state of the form:

$$\mathcal{N}^{\otimes n}(x_1^n(m_1), x_2^n(m_2)) \equiv \rho_{x_1^n(m_1), x_2^n(m_2)}^{B_1^n B_2^n} \in \mathcal{H}^{B_1^n B_2^n}. \quad (1)$$

To decode the message m_1 intended for him, Receiver 1 performs a positive operator-valued measure (POVM) $\{\Lambda_{m_1}\}_{m_1 \in \{1, \dots, |\mathcal{M}_1|\}}$ on the system B_1^n , the output of which we denote M'_1 . For all m_1 , Λ_{m_1} is a positive operator and $\sum_{m_1} \Lambda_{m_1} = I$. Receiver 2 similarly performs a POVM $\{\Gamma_{m_2}\}_{m_2 \in \{1, \dots, |\mathcal{M}_2|\}}$ on the system B_2^n , and the random variable associated with this outcome is denoted M'_2 .

An error event occurs whenever Receiver 1's measurement outcome is different from the message sent by Sender 1 ($M'_1 \neq m_1$) or Receiver 2's measurement outcome is different from the message sent by Sender 2 ($M'_2 \neq m_2$). The overall probability of error for message pair (m_1, m_2) is

$$p_e(m_1, m_2) \equiv \Pr \{ (M'_1, M'_2) \neq (m_1, m_2) \} \\ = \text{Tr} \left\{ (I - \Lambda_{m_1} \otimes \Gamma_{m_2}) \rho_{x_1^n(m_1), x_2^n(m_2)}^{B_1^n B_2^n} \right\},$$

where the measurement operator $(I - \Lambda_{m_1} \otimes \Gamma_{m_2})$ represents the complement of the correct decoding outcome.

Definition 3. An (n, R_1, R_2, ϵ) code for the interference channel consists of two codebooks $\{x_1^n(m_1)\}_{m_1 \in \mathcal{M}_1}$ and $\{x_2^n(m_2)\}_{m_2 \in \mathcal{M}_2}$, and two decoding POVMs $\{\Lambda_{m_1}\}_{m_1 \in \mathcal{M}_1}$ and $\{\Gamma_{m_2}\}_{m_2 \in \mathcal{M}_2}$, such that the average probability of error \bar{p}_e is bounded from above by ϵ :

$$\bar{p}_e \equiv \frac{1}{|\mathcal{M}_1| |\mathcal{M}_2|} \sum_{m_1, m_2} p_e(m_1, m_2) \leq \epsilon. \quad (2)$$

A rate pair (R_1, R_2) is *achievable* if there exists an $(n, R_1 - \delta, R_2 - \delta, \epsilon)$ quantum interference channel code for all $\epsilon, \delta > 0$ and sufficiently large n . The channel's *capacity region* is the closure of the set of all achievable rates.

IV. DECODING STRATEGIES FOR QUANTUM MULTIPLE ACCESS CHANNELS

The quantum interference channel described by $(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1, x_2}^{B_1 B_2}, \mathcal{H}^{B_1} \otimes \mathcal{H}^{B_2})$ induces two multiple access (MAC) sub-channels. More specifically MAC₁ is the channel to Receiver 1 given by $(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1, x_2}^{B_1} = \text{Tr}_{B_2} \{\rho_{x_1, x_2}^{B_1 B_2}\}, \mathcal{H}^{B_1})$, and MAC₂ is the channel to Receiver 2 defined by $(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1, x_2}^{B_2}, \mathcal{H}^{B_2})$. In order to better understand the interference channel problem we first consider the different decoding strategies for the individual receivers. In this section we analyze three types of decoding strategies for quantum multiple access channels, and then in Section V we use each of these to build a corresponding interference channel code.

Winter found a single-letter formula for the capacity of the classical-quantum multiple access channel [8].

Theorem 1 (Theorem 10 in [8]). *The capacity region for the classical-quantum multiple access channel $(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1, x_2}^B, \mathcal{H}^B)$ is given by*

$$C_{MAC} \equiv \text{conv} \bigcup_{p_{X_1}, p_{X_2}} \{(R_1, R_2) \in \mathbb{R}^2 \mid \text{Eqns. (3)-(5)}\}$$

$$R_1 \leq I(X_1; B|X_2)_\theta, \quad (3)$$

$$R_2 \leq I(X_2; B|X_1)_\theta, \quad (4)$$

$$R_1 + R_2 \leq I(X_1 X_2; B)_\theta, \quad (5)$$

where the information quantities are taken with respect to the classical-quantum state $\theta^{X_1 X_2 B}$ given by

$$\sum_{x_1, x_2} p_{X_1}(x_1) p_{X_2}(x_2) |x_1\rangle\langle x_1|^{X_1} \otimes |x_2\rangle\langle x_2|^{X_2} \otimes \rho_{x_1 x_2}^B. \quad (6)$$

A. Successive decoding

The technique used by Winter to prove the achievability of the rates in Theorem 1 is called *successive decoding*. For a given pair of probability distributions $p \equiv p_{X_1}, p_{X_2}$, the achievable rate region has the form of a pentagon bounded by the three inequalities in equations (3)-(5) and two rate positivity conditions. The two dominant vertices of this rate region have coordinates $\alpha_p \equiv (I(X_1; B)_\theta, I(X_2; B|X_1)_\theta)$ and $\beta_p \equiv (I(X_1; B|X_2)_\theta, I(X_2; B)_\theta)$ and correspond to two alternate successive decoding strategies.

To achieve the rates of α_p , the receiver first performs a measurement $\{\Lambda_{m_1}^\alpha\}$ to decode the message m_1 , and then performs a second measurement to recover the message m_2 . The second measurement is $\{\Lambda_{m_2|m_1}^\alpha\}$, where we have indicated that the second measurement is conditional on m_1 . By using these POVMs, Winter shows that if the rates (R_1, R_2) satisfy

$$R_1 \leq I(X_1; B)_\theta, \quad (7)$$

$$R_2 \leq I(X_2; B|X_1)_\theta, \quad (8)$$

then the expected success probability asymptotically approaches one:

$$\mathbb{E}_{X_1^n, X_2^n} \left\{ \frac{1}{|\mathcal{M}_1| |\mathcal{M}_2|} \sum_{m_1, m_2} \text{Tr} \{ \Lambda_{m_1, m_2}^w \rho_{X_1^n(m_1), X_2^n(m_2)} \} \right\} \geq 1 - \epsilon, \quad (9)$$

where we have informally denoted by Λ_{m_1, m_2}^w the successive measurements of $\Lambda_{m_1}^\alpha$ followed by $\Lambda_{m_2|m_1}^\alpha$.

The rate point β_p corresponds to the alternate decode ordering where the receiver decodes the message m_2 first and m_1 second. The corner points α_p and β_p are important because given codes that achieve them, we can use *time-sharing* and *resource wasting* to obtain all other rate pairs in the region. The M -sender MAC has $M!$ such corner points, one for each permutation of the decode ordering.

B. Quantum simultaneous decoding

Another approach for achieving the capacity of the multiple access channel, which does not use time-sharing, is simultaneous decoding. The analysis of the *classical* simultaneous decoder is a straightforward application of the joint

typicality lemma to bound the probability of the different decoding error events that may occur [10]. In the quantum case, we can similarly identify four different error events, but the construction of a measurement operator based on typical subspace projectors is more difficult to analyse because the different typical projectors may not commute in general.

In this section we prove that a quantum simultaneous decoder exists for multiple access channels with two senders and formulate Conjecture 3 regarding the existence of a simultaneous decoder for three-sender multiple access channels.

Theorem 2 (Two-sender simultaneous decoding). *Consider the cc-q multiple access channel with two senders and a single receiver $(\mathcal{X}_1 \times \mathcal{X}_2, \rho_{x_1, x_2}^B, \mathcal{H}^B)$. Let p_{X_i} be a distribution on \mathcal{X}_i and $\mathcal{M}_i \equiv \{1, \dots, 2^{n(R_i - \delta)}\}$ for $i \in \{1, 2\}$ and $\delta > 0$. Define the random codebooks $\{X_1^n(m_1)\}_{m_1 \in \mathcal{M}_1}$ and $\{X_2^n(m_2)\}_{m_2 \in \mathcal{M}_2}$ generated from the product distributions $p_{X_1^n}$ and $p_{X_2^n}$ respectively. There exists a simultaneous decoding POVM $\{\Lambda_{m_1, m_2}\}_{m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2}$, with expected average probability of error bounded from above by ϵ for all $\epsilon, \delta > 0$ and sufficiently large n provided the rates R_1, R_2 satisfy inequalities (3)-(5).*

The proof proceeds by random coding arguments using the properties of projectors onto the typical subspaces of the output states [11] and the *square-root* measurement. Note that Sen proved the same result using different techniques in [12].

Proof: Let state $\rho_{m_1, m_2} \equiv \rho_{x_1^n(m_1), x_2^n(m_2)}$ denote the output of the n uses of the channel when codewords $x_1^n(m_1)$ and $x_2^n(m_2)$ are input. Let $\Pi_{m_1, m_2}^n \equiv \Pi_{\rho_{x_1^n(m_1), x_2^n(m_2)}, \delta}^n$ be the conditionally typical projector for that state. Consider the following code-averaged output states:

$$\bar{\rho}_{x_1} \equiv \sum_{x_2} p_{X_2}(x_2) \rho_{x_1, x_2}, \quad (10)$$

$$\bar{\rho}_{x_2} \equiv \sum_{x_1} p_{X_1}(x_1) \rho_{x_1, x_2}, \quad (11)$$

$$\bar{\rho} \equiv \sum_{x_1, x_2} p_{X_1}(x_1) p_{X_2}(x_2) \rho_{x_1, x_2}. \quad (12)$$

Let $\Pi_{m_1}^n \equiv \Pi_{\bar{\rho}_{x_1^n(m_1)}, \delta}^n$ be the conditionally typical projector for the tensor product state $\bar{\rho}_{m_1} \equiv \bar{\rho}_{x_1^n(m_1)}$ defined by (10) for n uses of the channel. Let $\Pi_{m_2}^n \equiv \Pi_{\bar{\rho}_{x_2^n(m_2)}, \delta}^n$ be the conditionally typical projector for the tensor product state $\bar{\rho}_{m_2} \equiv \bar{\rho}_{x_2^n(m_2)}$ defined by (11) and finally let $\Pi_{\bar{\rho}, \delta}^n$ be the typical projector for the state $\bar{\rho}^{\otimes n}$ defined by (12).

The detection POVM $\{\Lambda_{m_1, m_2}\}$ has the following form:

$$\Lambda_{m_1, m_2} \equiv \left(\sum_{m'_1, m'_2} \Pi'_{m'_1, m'_2} \right)^{-\frac{1}{2}} \Pi'_{m_1, m_2} \left(\sum_{m'_1, m'_2} \Pi'_{m'_1, m'_2} \right)^{-\frac{1}{2}},$$

where

$$\Pi'_{m_1, m_2} \equiv \Pi_{\bar{\rho}, \delta}^n \cdot \Pi_{m_1}^n \cdot \Pi_{m_2}^n, \quad (13)$$

is a positive operator which consists of three typical projectors “sandwiched” together.

The average error probability of the code is given by:

$$\bar{p}_e \equiv \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{m_1, m_2} \text{Tr} \{ (I - \Lambda_{m_1, m_2}) \rho_{m_1, m_2} \}. \quad (14)$$

One key insight for the proof is the substitution of the output state ρ_{m_1, m_2} with a smoothed version:

$$\tilde{\rho}_{m_1, m_2} \equiv \Pi_{m_2}^n \rho_{m_1, m_2} \Pi_{m_2}^n, \quad (15)$$

and bounding (14) from above as follows:

$$\bar{p}_e \leq \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{m_1, m_2} \left[\text{Tr} \{ (I - \Lambda_{m_1, m_2}) \tilde{\rho}_{m_1, m_2} \} + \|\tilde{\rho}_{m_1, m_2} - \rho_{m_1, m_2}\|_1 \right]. \quad (16)$$

To obtain (16), we used the inequality

$$\text{Tr} \{ \Lambda \rho \} \leq \text{Tr} \{ \Lambda \sigma \} + \|\rho - \sigma\|_1, \quad (17)$$

which holds for all operators such that $0 \leq \rho, \sigma, \Lambda \leq I$.

The Hayashi-Nagaoka operator inequality applies to all positive operators T and S where $0 \leq S \leq I$ [13]:

$$I - (S + T)^{-\frac{1}{2}} S (S + T)^{-\frac{1}{2}} \leq 2(I - S) + 4T.$$

Choosing $S = \Pi'_{m_1, m_2}$, $T = \sum_{(m'_1, m'_2) \neq (m_1, m_2)} \Pi'_{m'_1, m'_2}$, we apply the above operator inequality to bound the average error probability of the first term in (16) as:

$$\begin{aligned} \bar{p}_e \leq & \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{m_1, m_2} \left[2\text{Tr} \{ (I - \Pi'_{m_1, m_2}) \tilde{\rho}_{m_1, m_2} \} \right. \\ & \left. + 4 \sum_{(m'_1, m'_2) \neq (m_1, m_2)} \text{Tr} \{ \Pi'_{m'_1, m'_2} \tilde{\rho}_{m_1, m_2} \} + \|\tilde{\rho}_{m_1, m_2} - \rho_{m_1, m_2}\|_1 \right]. \end{aligned} \quad (18)$$

We apply a random coding argument to bound the expectation of the average error probability in (18). A bound on the first term follows from the following argument:

$$\begin{aligned} & \mathbb{E}_{X_1^n, X_2^n} \text{Tr} \{ \Pi'_{m_1, m_2} \tilde{\rho}_{m_1, m_2} \} \\ &= \mathbb{E}_{X_1^n, X_2^n} \text{Tr} \{ \Pi_{\bar{\rho}, \delta}^n \cdot \Pi_{m_1}^n \cdot \Pi_{m_1, m_2}^n \Pi_{m_2}^n \rho_{m_1, m_2} \Pi_{m_2}^n \} \\ &\geq \mathbb{E}_{X_1^n, X_2^n} \text{Tr} \{ \Pi_{m_1, m_2}^n \rho_{m_1, m_2} \} \\ &\quad - \mathbb{E}_{X_1^n, X_2^n} \left\| \Pi_{m_2}^n \rho_{m_1, m_2} \Pi_{m_2}^n - \rho_{m_1, m_2} \right\|_1 \\ &\quad - \mathbb{E}_{X_1^n, X_2^n} \left\| \Pi_{\bar{\rho}, \delta}^n \rho_{m_1, m_2} \Pi_{\bar{\rho}, \delta}^n - \rho_{m_1, m_2} \right\|_1 \\ &\quad - \mathbb{E}_{X_1^n, X_2^n} \left\| \Pi_{m_1}^n \rho_{m_1, m_2} \Pi_{m_1}^n - \rho_{m_1, m_2} \right\|_1 \\ &\geq 1 - \epsilon - 6\sqrt{\epsilon}. \end{aligned} \quad (19)$$

The first inequality follows from (17) applied three times. The second inequality follows from the *Gentle Measurement Lemma* for ensembles [11, Lemma 9.4.3] and the properties of entropy-typical projectors [11, Section 14.2.2]. The same reasoning is used to obtain a bound the expectation of the smoothing-penalty term in equation (18): $\mathbb{E}_{X_1^n, X_2^n} \|\tilde{\rho}_{m_1, m_2} - \rho_{m_1, m_2}\|_1 \leq 2\sqrt{\epsilon}$.

We decompose the second term in (18) into three error events, each representing a different type of decoding error:

$$\begin{aligned} & \sum_{(m'_1, m'_2) \neq (m_1, m_2)} \text{Tr} \{ \Pi'_{m'_1, m'_2} \tilde{\rho}_{m_1, m_2} \} = \\ &= \sum_{m'_1 \neq m_1} \text{Tr} \{ \Pi'_{m'_1, m_2} \tilde{\rho}_{m_1, m_2} \} \quad (E1) \\ &\quad + \sum_{m'_2 \neq m_2} \text{Tr} \{ \Pi'_{m_1, m'_2} \tilde{\rho}_{m_1, m_2} \} \quad (E2) \\ &\quad + \sum_{m'_1 \neq m_1, m'_2 \neq m_2} \text{Tr} \{ \Pi'_{m'_1, m'_2} \tilde{\rho}_{m_1, m_2} \}. \quad (E12) \end{aligned}$$

The expectation the over random choice of codebook for event (E1), the event that m_1 is decoded incorrectly, is as follows:

$$\begin{aligned} & \mathbb{E}_{X_1^n, X_2^n} \left\{ \sum_{m'_1 \neq m_1} \text{Tr} \left[\Pi'_{m'_1, m_2} \tilde{\rho}_{m_1, m_2} \right] \right\} \\ &= \sum_{m'_1 \neq m_1} \mathbb{E}_{X_2^n} \left\{ \text{Tr} \left[\mathbb{E}_{X_1^n} \{ \Pi'_{m'_1, m_2} \} \mathbb{E}_{X_1^n} \{ \tilde{\rho}_{m_1, m_2} \} \right] \right\} \\ &= \sum_{m'_1 \neq m_1} \mathbb{E}_{X_1^n, X_2^n} \left\{ \text{Tr} \left[\Pi'_{m'_1, m_2} \Pi_{m_2}^n \tilde{\rho}_{m_2} \Pi_{m_2}^n \right] \right\} \\ &\leq 2^{-n[H(B|X_2) - \delta]} \sum_{m'_1 \neq m_1} \mathbb{E}_{X_1^n, X_2^n} \left\{ \text{Tr} \left[\Pi'_{m'_1, m_2} \Pi_{m_2}^n \right] \right\} \end{aligned}$$

The first equality follows because the codewords labeled by m'_1 and m_1 are independent. The second equality comes from the definition of the averaged code state $\bar{\rho}_{m_2} \equiv \bar{\rho}_{x_2^n}(m_2)$. The last inequality follows from:

$$\Pi_{m_2}^n \bar{\rho}_{m_2} \Pi_{m_2}^n \leq 2^{-n[H(B|X_2) - \delta]} \Pi_{m_2}^n.$$

We focus our attention on the expression inside the trace:

$$\begin{aligned} & \text{Tr} \left[\Pi'_{m'_1, m_2} \Pi_{m_2}^n \right] \\ &= \text{Tr} \left[\Pi_{\bar{\rho}, \delta}^n \cdot \Pi_{m'_1}^n \cdot \Pi_{m'_1, m_2}^n \Pi_{m_2}^n \right] \\ &= \text{Tr} \left[\Pi_{\bar{\rho}, \delta}^n \Pi_{m'_1}^n \Pi_{m'_1, m_2}^n \Pi_{m_1}^n \Pi_{\bar{\rho}, \delta}^n \Pi_{m_2}^n \right] \\ &= \text{Tr} \left[\Pi_{m'_1}^n \Pi_{\bar{\rho}, \delta}^n \Pi_{m_2}^n \Pi_{\bar{\rho}, \delta}^n \Pi_{m'_1}^n \Pi_{m'_1, m_2}^n \right] \\ &\leq \text{Tr} \left[\Pi_{m'_1}^n \right]. \end{aligned}$$

In the first step we substituted the definition of Π'_{m_1, m_2} from equation (13). The rest of the equalities follow from the cyclicity of trace. The inequality follows from

$$\Pi_{m'_1}^n \Pi_{\bar{\rho}, \delta}^n \Pi_{m_2}^n \Pi_{\bar{\rho}, \delta}^n \Pi_{m'_1}^n \leq \Pi_{m'_1}^n \Pi_{\bar{\rho}, \delta}^n \Pi_{m'_1}^n \leq \Pi_{m'_1}^n \leq I. \quad (20)$$

Continuing, we obtain the following bound on the expected probability of error event (E1):

$$\begin{aligned} & \mathbb{E}_{X_1^n, X_2^n} \{ (E1) \} \leq 2^{-n[H(B|X_2) - \delta]} \sum_{m'_1 \neq m_1} \mathbb{E}_{X_1^n, X_2^n} \left\{ \text{Tr} \left\{ \Pi_{m'_1, m_2}^n \right\} \right\} \\ &\leq 2^{-n[H(B|X_2) - \delta]} \sum_{m'_1 \neq m_1} 2^{n[H(B|X_1, X_2) + \delta]} \\ &\leq |\mathcal{M}_1| 2^{-n[I(X_1; B|X_2) - 2\delta]}. \end{aligned} \quad (21)$$

The second inequality in (21) follows from the bound

$$\text{Tr} \{ \Pi_{m_1, m_2}^n \} \leq 2^{n[H(B|X_1, X_2) + \delta]}$$

on the rank of a conditionally typical projector.

We employ a different argument to bound the probability of the second error event (E2) based on the following fact

$$\begin{aligned}\Pi_{m_1, m_2}^n &\leq 2^{n[H(B|X_1 X_2)+\delta]} \Pi_{m_1, m_2}^n \rho_{m_1, m_2}^B \Pi_{m_1, m_2}^n \\ &= 2^{n[H(B|X_1 X_2)+\delta]} \sqrt{\rho_{m_1, m_2}^B \Pi_{m_1, m_2}^n} \sqrt{\rho_{m_1, m_2}^B} \\ &\leq 2^{n[H(B|X_1 X_2)+\delta]} \rho_{m_1, m_2}^B,\end{aligned}\quad (22)$$

which we refer to as the *projector trick* [14]. The first inequality is the standard lower bound on the eigenvalues of ρ_{m_1, m_2}^B expressed as an operator upper bound on the projector Π_{m_1, m_2}^n . The equality follows because the state and its typical projector commute. The last inequality follows from $0 \leq \Pi_{m_1, m_2}^n \leq I$.

Continuing,

$$\begin{aligned}\mathbb{E}_{X_1^n, X_2^n} \{(\text{E2})\} &= \mathbb{E}_{X_1^n, X_2^n} \left\{ \sum_{m_2' \neq m_2} \text{Tr} \left[\Pi_{m_1, m_2'}^n \tilde{\rho}_{m_1, m_2} \right] \right\} \\ &= \sum_{m_2' \neq m_2} \mathbb{E}_{X_1^n} \left\{ \text{Tr} \left[\mathbb{E}_{X_2^n} \left\{ \Pi_{m_1, m_2'}^n \right\} \mathbb{E}_{X_2^n} \left\{ \tilde{\rho}_{m_1, m_2} \right\} \right] \right\} \\ &= \sum_{m_2' \neq m_2} \mathbb{E}_{X_1^n} \left\{ \text{Tr} \left[\mathbb{E}_{X_2^n} \left\{ \Pi_{\tilde{\rho}, \delta}^n \cdot \Pi_{m_1}^n \cdot \Pi_{m_1, m_2'}^n \right\} \mathbb{E}_{X_2^n} \left\{ \tilde{\rho}_{m_1, m_2} \right\} \right] \right\} \\ &= \sum_{m_2' \neq m_2} \mathbb{E}_{X_1^n} \left\{ \text{Tr} \left[\mathbb{E}_{X_2^n} \left\{ \Pi_{\tilde{\rho}, \delta}^n \mathbb{E}_{X_2^n} \left\{ \Pi_{m_1}^n \Pi_{m_1, m_2'}^n \Pi_{m_1}^n \right\} \Pi_{\tilde{\rho}, \delta}^n \right\} \mathbb{E}_{X_2^n} \left\{ \tilde{\rho}_{m_1, m_2} \right\} \right] \right\}\end{aligned}$$

We focus our attention on the first expectation term:

$$\begin{aligned}\mathbb{E}_{X_2^n} \left\{ \Pi_{m_1}^n \Pi_{m_1, m_2'}^n \Pi_{m_1}^n \right\} &\leq 2^{n[H(B|X_1 X_2)+\delta]} \mathbb{E}_{X_2^n} \left\{ \Pi_{m_1}^n \rho_{m_1, m_2'}^B \Pi_{m_1}^n \right\} \\ &= 2^{n[H(B|X_1 X_2)+\delta]} \Pi_{m_1}^n \mathbb{E}_{X_2^n} \left\{ \rho_{m_1, m_2'}^B \right\} \Pi_{m_1}^n \\ &= 2^{n[H(B|X_1 X_2)+\delta]} \Pi_{m_1}^n \bar{\rho}_{m_1} \Pi_{m_1}^n \\ &\leq 2^{n[H(B|X_1 X_2)+\delta]} 2^{-n[H(B|X_1)-\delta]} \Pi_{m_1}^n \\ &= 2^{-n[I(X_2; B|X_1)-2\delta]} \Pi_{m_1}^n\end{aligned}$$

Substituting back into the expression for the error bound, we obtain:

$$\begin{aligned}\mathbb{E}_{X_1^n, X_2^n} \{(\text{E2})\} &\leq 2^{-n[I(X_2; B|X_1)-2\delta]} \sum_{m_2' \neq m_2} \text{Tr} \left[\Pi_{\tilde{\rho}, \delta}^n \Pi_{m_1}^n \Pi_{\tilde{\rho}, \delta}^n \tilde{\rho}_{m_1, m_2} \right] \\ &= 2^{-n[I(X_2; B|X_1)-2\delta]} \sum_{m_2' \neq m_2} \text{Tr} \left[\Pi_{\tilde{\rho}, \delta}^n \Pi_{m_1}^n \Pi_{\tilde{\rho}, \delta}^n \Pi_{m_2}^n \rho_{m_1, m_2} \Pi_{m_2}^n \right] \\ &= 2^{-n[I(X_2; B|X_1)-2\delta]} \sum_{m_2' \neq m_2} \text{Tr} \left[\Pi_{m_2}^n \Pi_{\tilde{\rho}, \delta}^n \Pi_{m_1}^n \Pi_{\tilde{\rho}, \delta}^n \Pi_{m_2}^n \rho_{m_1, m_2} \right] \\ &\leq 2^{-n[I(X_2; B|X_1)-2\delta]} \sum_{m_2' \neq m_2} \text{Tr} \left[\rho_{m_1, m_2} \right] \\ &\leq 2^{-n[I(X_2; B|X_1)-2\delta]} |\mathcal{M}_2|,\end{aligned}\quad (23)$$

The second inequality follows from an argument analogous to (20).

By a different argument involving averaged states, we bound the probability of the third error event as:

$$\mathbb{E}_{X_1^n, X_2^n} \left\{ (\text{E12}) \right\} \leq |\mathcal{M}_1| |\mathcal{M}_2| 2^{-n[I(X_1 X_2; B)-2\delta]}.\quad (24)$$

Combining the bounds from equations (19), (21), (23), (24) and the smoothing penalty, we get the following bound on the expectation of the average error probability:

$$\begin{aligned}\mathbb{E}_{X_1^n, X_2^n} \left\{ \bar{p}_e \right\} &\leq 2(\epsilon + 6\sqrt{\epsilon}) + 2\sqrt{\epsilon} \\ &+ 4 \left[|\mathcal{M}_1| 2^{-n[I(X_1; B|X_2)-2\delta]} + |\mathcal{M}_2| 2^{-n[I(X_2; B|X_1)-2\delta]} \right. \\ &\quad \left. + |\mathcal{M}_1| |\mathcal{M}_2| 2^{-n[I(X_1 X_2; B)-2\delta]} \right].\end{aligned}$$

Thus, if we choose the message sets sizes to be $|\mathcal{M}_1| = 2^{n[R_1-3\delta]}$, and $|\mathcal{M}_2| = 2^{n[R_2-3\delta]}$, the expectation of the average error probability vanishes whenever the rates R_1 and R_2 obey the inequalities:

$$\begin{aligned}R_1 - \delta &< I(X_1; B|X_2), \quad R_2 - \delta < I(X_2; B|X_1), \\ R_1 + R_2 - 4\delta &< I(X_1 X_2; B).\end{aligned}$$

Given that $\delta > 0$ is an arbitrarily small number the bounds in the statement of the theorem follow. ■

We now state our conjecture regarding the existence of a quantum simultaneous decoder for the three-sender case.

Conjecture 3 (Three-sender QMAC simultaneous decoding). *Let $\mathcal{C}_{3\text{MAC}}$ denote the capacity region of a ccc-q multiple access channel with three senders: $x_1, x_2, x_3 \rightarrow \rho_{x_1, x_2, x_3}^B$. Let $\{X_i^n(m_i)\}_{m_i \in \mathcal{M}_i}$, for $i \in \{1, 2, 3\}$ be random codebooks generated according to the product distributions $p_{X_i^n}^n$ with messages sets $\mathcal{M}_i \equiv \{1, \dots, 2^{n(R_i - \delta)}\}$ with $\delta > 0$. There exists a simultaneous decoding POVM $\{\Lambda_{m_1, m_2, m_3}\}$, with expected average probability of error bounded from above by ϵ for all $\epsilon, \delta > 0$ and sufficiently large n for any rate triple $(R_1, R_2, R_3) \in \mathcal{C}_{3\text{MAC}}$.*

Were this conjecture true, it would form a fundamental building block for multiuser information theory. Obtaining a proof might allow us to directly adapt many of the known classical techniques of classical multiuser information theory to the quantum setting. Indeed, many coding theorems in classical network information theory exploit a simultaneous decoding approach (jointly typical decoding) [10].

We can prove that simultaneous decoding works for a special class of three-sender MACs for which the averaged output states (defined analogously to (10) and (11)) satisfy the following commutation relations: $[\bar{\rho}_{x_1, x_2}, \bar{\rho}_{x_2, x_3}] = 0$, $[\bar{\rho}_{x_1, x_3}, \bar{\rho}_{x_1, x_2}] = 0$, $[\bar{\rho}_{x_1, x_3}, \bar{\rho}_{x_2, x_3}] = 0$, $\forall x_1, x_2, x_3$. These commutation relations imply that the corresponding typical projectors commute and thus give a simpler construction of the measurement operator.

Furthermore, we can prove that a quantum simultaneous decoder exists for a random code provided that the rates R_1 , R_2 and R_3 satisfy a set of stronger constraints involving min-entropies. We invite the reader to consult [15] for further details about these special cases.

C. Rate-splitting

Rate-splitting is another approach for achieving the classical multiple access channel rate region [9], which generalizes

readily to the quantum case using the successive decoding approach in [8].

Lemma 1. *For a given $p = p_{X_1}, p_{X_2}$, any rate pair (R_1, R_2) that lies in between the two corner points of the MAC rate region α_p and β_p can be achieved if Sender 2 splits her message m_2 into two parts m_{2u} and m_{2v} and encodes them with a split codebook $(\{u^n(m_{2u})\}_{m_{2u}}, \{v^n(m_{2v})\}_{m_{2v}}, f)$. The receiver decodes the messages in the order $m_{2u} \rightarrow m_1 \rightarrow m_{2v}$ using successive decoding.*

The rate-split codebook consists of two random codebooks generated from p_U and p_V and a mixing function such that $f(U, V) = X_2$. For a fixed rate pair (R_1, R_2) , the construction of a split codebook achieving this rate pair depends on the properties of the channel for which we are coding.

V. QUANTUM INTERFERENCE CHANNELS

In this section we calculate achievable rate regions for the quantum interference channel based on three decoding strategies: successive decoding, simultaneous decoding and rate-splitting. We also show the quantum Han-Kobayashi inner bound, which relies on Conjecture 3 for its proof.

A. Rates achievable by successive decoding

In this section, we require the receivers to decode the messages of both senders. Let the decoding ordering of Receiver 1 be represented by a permutation $\pi_1: \pi_1 = (1, 2)$ when decoding in the order $m_1 \rightarrow m_2$, and $\pi_1 = (2, 1)$ for the alternate decoding order. We similarly let $\pi_2 = (1, 2)$ and $\pi_2 = (2, 1)$ denote the two decode orderings for Receiver 2. If we use a successive decoding strategy at both receivers, and calculate the best possible rates that are compatible with both receivers' ability to decode, we obtain an achievable rate region. Consider, for example, the decoding strategy $\pi_1 = (2, 1), \pi_2 = (2, 1)$, which corresponds to both receivers decoding in the order $m_2 \rightarrow m_1$. In this case, we know that the code is decodable for Receiver 1 provided $R_1 < I(X_1; B_1|X_2)$ and $R_2 < I(X_2; B_1)$. Receiver 2 will be able to decode provided $R_2 < I(X_2; B_2)$ (we do not require Receiver 2 to decode m_1 after he has decoded m_2).

Thus, the rate pair $R_1 < I(X_1; B_1|X_2)$, $R_2 < \min\{I(X_2; B_1), I(X_2; B_2)\}$ is achievable for the interference channel. Similarly, for all possible pairs of permutations π_1, π_2 , we obtain an achievable rate pair for the interference channel.

For interference channels with "very strong" interference [3], such that for all input distributions p_{X_1} and p_{X_2} ,

$$I(X_1; B_1|X_2) \leq I(X_1; B_2), \quad (25)$$

$$I(X_2; B_2|X_1) \leq I(X_2; B_1), \quad (26)$$

the rates achieved by the successive decoding strategy $\pi_1 = (2, 1), \pi_2 = (1, 2)$ are optimal.

Theorem 4 (Channels with very strong interference). *The channel's capacity region is the union of all rates R_1 and*

R_2 satisfying the inequalities:

$$R_1 \leq I(X_1; B_1|X_2Q)_\theta,$$

$$R_2 \leq I(X_2; B_2|X_1Q)_\theta,$$

with union taken over input distributions $p_Q, p_{X_1|Q}$ and $p_{X_2|Q}$.

The matching outer bound follows from the converse part of Theorem 1, since the individual rates are optimal in the two MAC sub-channels [3]. Indeed, we can pursue the connection between the IC and the MAC sub-channels further to obtain a simple outer bound for the capacity of general quantum interference channels analogous to the classical result by Sato [1].

Theorem 5. *Consider the Sato region defined as follows:*

$$\mathcal{R}_{\text{Sato}}(\mathcal{N}) \triangleq \bigcup_{p \in \mathcal{P}_{\text{Sato}}} \{(R_1, R_2)\}, \quad (27)$$

where R_1 and R_2 are rates satisfying the inequalities:

$$R_1 \leq I(X_1; B_1|X_2Q)_\theta, \quad (28)$$

$$R_2 \leq I(X_2; B_2|X_1Q)_\theta, \quad (29)$$

$$R_1 + R_2 \leq I(X_1X_2; B_1B_2|Q)_\theta, \quad (30)$$

where the union is taken over over all input distributions of the form $p_Q(q) p_{X_1|Q}(x_1|q) p_{X_2|Q}(x_2|q)$ and the resulting average input-output state θ . Then the region $\mathcal{R}_{\text{Sato}}$ is an outer bound on the capacity region of the general quantum interference channel.

This proof follows from the observation that any code for the quantum interference channel also gives codes for three quantum multiple access channel subproblems: one for Receiver 1, another for Receiver 2, and a third for the two receivers considered together. Thus, using the outer bound on the quantum multiple access channel rates from Theorem 1 we obtain the outer bound in Theorem 5.

B. Rates achievable by two-sender simultaneous decoding

The simultaneous decoder from Theorem 2 allows us to calculate the capacity region for quantum interference channels with "strong" interference [16], [17], for which the following condition holds:

$$I(X_1; B_1|X_2) \leq I(X_1; B_2|X_2), \quad (31)$$

$$I(X_2; B_2|X_1) \leq I(X_2; B_1|X_1), \quad (32)$$

for all input distributions p_{X_1} and p_{X_2} .

Theorem 6 (Channels with strong interference). *The channel's capacity region is the union of all rates R_1 and R_2 satisfying the inequalities:*

$$R_1 \leq I(X_1; B_1|X_2Q)_\theta,$$

$$R_2 \leq I(X_2; B_2|X_1Q)_\theta,$$

$$R_1 + R_2 \leq \min\{I(X_1X_2; B_1|Q), I(X_1X_2; B_2|Q)\}_\theta,$$

where the union is over input distributions $p_{X_1|Q} p_{X_2|Q} p_Q$.

This rate region describes the intersection of the MAC rate regions for the two receivers and corresponds to the condition that we require each receiver to decode both m_1 and m_2 .

C. The quantum Han-Kobayashi rate region

For general interference channels the Han-Kobayashi coding strategy gives the best known achievable rate region [4] and involves *partial* decoding of the interfering signal. Instead of using a standard codebook $\{x_1^n(m_1)\}_{m_1 \in \mathcal{M}_1}$ at a rate $R_1 \equiv \frac{1}{n} \log |\mathcal{M}_1|$ to encode her message m_1 , Sender 1 splits her message into two parts: a *personal* message m_{1p} encoded using a random codebook $\{u_1^n(m_{1p})\}_{m_{1p} \in \mathcal{M}_{1p}}$ and a common message m_{1c} encoded into $\{w_1^n(m_{1c})\}_{m_{1c} \in \mathcal{M}_{1c}}$. In terms of rates, this means that the sum rate $R_{1p} + R_{1c}$ should be equal to the original rate R_1 . So long as Receiver 1 can decode both parts m_{1p} and m_{1c} , he can reconstruct the original message m_1 . Receiver 2 can decode Sender 1's common message m_{1c} and improve his communication rate to $R_2 = I(X_2; B_2 | W_1)$ by using this side information. To return the favor, Sender 2 also splits her message into m_{2p} and m_{2c} . The overall codebook is generated from the class of Han-Kobayashi probability distributions, \mathcal{P}_{HK} , which factorize as $p(q)p(u_1|q)p(w_1|q)p(x_1|u_1, w_1)p(u_2|q)p(w_2|q)p(x_2|u_2, w_2)$, where $p(x_1|u_1, w_1)$ and $p(x_2|u_2, w_2)$ are degenerate probability distributions that correspond to deterministic functions f_1 and f_2 , $f_i: \mathcal{U}_i \times \mathcal{W}_i \rightarrow \mathcal{X}_i$, which are used to combine symbols of U and W to produce a symbol X suitable as input to the channel.

Theorem 7. *The quantum Han-Kobayashi rate region:*

$$\mathcal{R}_{HK} \equiv \bigcup_{p \in \mathcal{P}_{HK}} \{(R_1, R_2) \in \mathbb{R}^2 \mid \text{Eqns. (HK1) - (HK9)}\}$$

$$R_1 \leq I(U_1 W_1; B_1 | W_2 Q) \quad (\text{HK1})$$

$$R_1 \leq I(U_1; B_1 | W_1 W_2 Q) + I(W_1; B_2 | U_2 W_2 Q) \quad (\text{HK2})$$

$$R_2 \leq I(U_2 W_2; B_2 | W_1 Q) \quad (\text{HK3})$$

$$R_2 \leq I(W_2; B_1 | U_1 W_1 Q) + I(U_2; B_2 | W_1 W_2 Q) \quad (\text{HK4})$$

$$R_1 + R_2 \leq I(U_1 W_1 W_2; B_1 | Q) + I(U_2; B_2 | W_1 W_2 Q) \quad (\text{HK5})$$

$$R_1 + R_2 \leq I(U_1; B_1 | W_2 W_1 Q) + I(U_2 W_2 W_1; B_2 | Q) \quad (\text{HK6})$$

$$R_1 + R_2 \leq I(U_1 W_2; B_1 | W_1 Q) + I(U_2 W_1; B_2 | W_2 Q) \quad (\text{HK7})$$

$$2R_1 + R_2 \leq I(U_1; B_1 | W_1 W_2 Q) + I(U_2 W_1; B_2 | W_2 Q) + I(U_1 W_1 W_2; B_1 | Q) \quad (\text{HK8})$$

$$R_1 + 2R_2 \leq I(U_1 W_2; B_1 | W_1 Q) + I(U_2; B_2 | W_2 W_1 Q) + I(U_2 W_2 W_1; B_2 | Q) \quad (\text{HK9})$$

where the information theoretic quantities are taken with respect to a state of the form:

$$\sum_{q, u_1, u_2, w_1, w_2} p_Q(q) p_{U_1}(u_1|q) p_{U_2}(u_2|q) p_{W_1}(w_1|q) p_{W_2}(w_2|q) |q\rangle\langle q|^Q \otimes |u_1\rangle\langle u_1|^{U_1} \otimes |u_2\rangle\langle u_2|^{U_2} \otimes |w_1\rangle\langle w_1|^{W_1} \otimes |w_2\rangle\langle w_2|^{W_2} \otimes \rho_{f_1(u_1, w_1), f_2(u_2, w_2)}^{B_1 B_2} \quad (33)$$

is an achievable rate region provided Conjecture 3 holds.

The proof is in the same spirit as the original result of Han and Kobayashi [4]. Our result is conditional on Conjecture 3 for the construction of the decoding POVMs: $\{\Lambda_{m_{1p}, m_{1c}, m_{2c}}\}$ for Receiver 1, and $\{\Gamma_{m_{1c}, m_{2c}, m_{2p}}\}$ for Receiver 2. Refer to [15] for the proof.

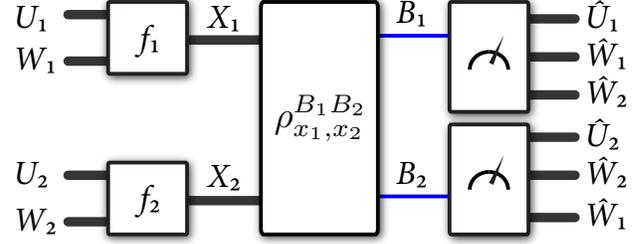


Fig. 1. The Han-Kobayashi coding strategy. Sender 1 selects codewords according to a “personal” random variable U_1 and a “common” random variable W_1 . She then acts on U_1 and W_1 with some deterministic function f_1 that outputs a variable X_1 which serves as a classical input to the interference channel. Sender 2 uses a similar encoding. Receiver 1 performs a measurement to decode both variables of Sender 1 and the common random variable W_2 of Sender 2. Receiver 2 acts similarly.

D. Using rate-splitting for the IC

We can use rate-splitting to improve the successive decoding region described in Section V-A. Inspired by the Han-Kobayashi strategy we make the senders split their messages into two parts: $m_1 \rightarrow m_{1p}, m_{1c}$ and $m_2 \rightarrow m_{2p}, m_{2c}$. Such a split induces two three-user multiple access channels. Receiver 1 decodes the messages m_{1p}, m_{1c} and m_{2c} using successive decoding, and there are six different decode orderings he can use. We can naturally use all 6×6 pairs of decoding orders to obtain a set of achievable rate pairs.

Proposition 8. *Consider the rate point P associated with the decode ordering π_1 for Receiver 1 and π_2 for Receiver 2:*

$$P = \left(R_{1p}^{(1)} + \min\{R_{1c}^{(1)}, R_{1c}^{(2)}\}, \min\{R_{2c}^{(1)}, R_{2c}^{(2)}\} + R_{2p}^{(2)} \right),$$

where the rates constraints for Receiver j satisfy

$$R_{\pi_j(1)}^{(j)} \leq I(X_{\pi_j(1)}; B_j), \quad (34)$$

$$R_{\pi_j(2)}^{(j)} \leq I(X_{\pi_j(2)}; B_j | X_{\pi_j(1)}), \quad (35)$$

$$R_{\pi_j(3)}^{(j)} \leq I(X_{\pi_j(3)}; B_j | X_{\pi_j(1)} X_{\pi_j(2)}). \quad (36)$$

The rate pair P is achievable for the quantum interference channel, for all permutations π_1 of the set of indices $(1p, 1c, 2c)$ and for all permutations π_2 of the set $(2p, 2c, 1c)$.

The rate region described by the convex hull of the points P is generally smaller than the Han-Kobayashi region as illustrated in Figure 2. An interesting open problem is whether we can achieve all rates of the Han-Kobayashi region by splitting each sender's message into more than two parts and using only rate-splitting [9] and successive decoding. There exists an attempt to answer this question for the classical interference channel [18]. The argument in that paper is based on a careful analysis of the geometrical structure of the Chong-Motani-Garg region, which is known to be equivalent to the Han-Kobayashi region when considering all possible input distributions [19]. An implicit assumption is made that the change of the code distribution dictated by applying the rate-splitting technique at the convenience of one receiver

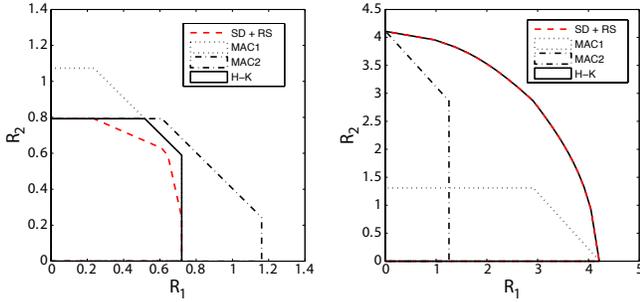


Fig. 2. These two figures plot rate pairs that are achievable with successive decoding and rate-splitting (SD+RS). The figures compare these rates with those achievable by the Han-Kobayashi (HK) coding strategy, while also plotting the regions corresponding to the two induced multiple access channels to each receiver (MAC1 and MAC2). The LHS figure demonstrates that SD+RS does not in general achieve the full capacity region for channels with strong interference. For this case we can use the two-sender simultaneous decoder from Theorem 2. The RHS figure demonstrates that, for some channels with weak interference, SD+RS is virtually indistinguishable from HK.

does not affect the other receiver’s decoding ability. Unfortunately, this assumption does not hold in general, which can be seen from the following argument.

Consider a code for an interference channel where the message $m_1 \in \{1, \dots, 2^{nR}\}$ is to be decoded by both receivers. Suppose we have $R = I(X_1; Y_2)$ and $R \leq I(X_1; Y_1)$ for some input distribution p_{X_1} . If we generate a standard random codebook of size 2^{nR} , then both receivers will be able to decode the message encoded in X_1 . However, we might want to use a split codebook generated according to distributions p_U and p_V , and the mixing function $f(U, V) = X_1$. If we generate the split codebook for Receiver 2 then we should pick the rate $R_U = I(U; Y_2)$ so that Receiver 2 will be able to decode U with small error probability. We should however keep in mind that we are coding for an interference channel and we also want Receiver 1 to decode X_1 . The problem is that it is possible that $R_U > I(U; Y_1)$, in which case Receiver 1 cannot decode U and thus cannot decode the message by successive decoding. In this case, the code obtained by splitting according to the second receiver’s prescription is not a good code for the interference channel.

VI. DISCUSSION

There are several open questions regarding this work. First, we would of course like to prove Conjecture 3 holds because it would be a powerful building block for multi-user quantum Shannon theory. Also, we would like to study the channel’s quantum, entanglement-assisted, and hybrid classical-quantum capacities. Finally, it could be that three-sender quantum simultaneous decoding is not necessary for achieving the Han-Kobayashi region. If the classical Han-Kobayashi rate region for the discrete memoryless interference channel can be achieved using rate-splitting and successive decoding, then this would be another way to prove Theorem 7 without appealing to Conjecture 3.

We acknowledge discussions with Frédéric Dupuis, Eren

Şaşoğlu and Mai Vu. P. Hayden acknowledges support from the Canada Research Chairs program, the Perimeter Institute, CIFAR, FQRNT’s INTRIQ, MITACS, NSERC, ONR through grant N000140811249, and QuantumWorks. M. M. Wilde acknowledges support from the MDEIE (Québec) PSR-SIIRI international collaboration grant. I. Savov acknowledges support from FQRNT and NSERC.

REFERENCES

- [1] H. Sato, “Two-user communication channels,” *IEEE Transactions on Information Theory*, vol. 23, no. 3, pp. 295–304, 1977.
- [2] A. B. Carleial, “Interference channels,” *IEEE Transactions on Information Theory*, vol. 24, no. 1, pp. 60–70, 1978.
- [3] —, “A case where interference does not reduce capacity,” *IEEE Transactions on Information Theory*, vol. 21, p. 569, 1975.
- [4] T. S. Han and K. Kobayashi, “A new achievable rate region for the interference channel,” *IEEE Transactions on Information Theory*, vol. 27, no. 1, pp. 49–60, January 1981.
- [5] S. Guha, I. Savov, and M. M. Wilde, “The free space optical interference channel,” *accepted for Proceedings of the International Symposium on Information Theory, St. Petersburg, Russia*, August 2011, arXiv:1102.2627.
- [6] A. S. Holevo, “The capacity of the quantum channel with general signal states,” *IEEE Transactions on Information Theory*, vol. 44, no. 1, pp. 269–273, 1998.
- [7] B. Schumacher and M. D. Westmoreland, “Sending classical information via noisy quantum channels,” *Physical Review A*, vol. 56, no. 1, pp. 131–138, July 1997.
- [8] A. Winter, “The capacity of the quantum multiple-access channel,” *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 3059–3065, 2001.
- [9] A. J. Grant, B. Rimoldi, R. L. Urbanke, and P. A. Whiting, “Rate-splitting multiple access for discrete memoryless channels,” *IEEE Transactions on Information Theory*, vol. 47, no. 3, pp. 873–890, 2001.
- [10] A. El Gamal and Y.-H. Kim, “Lecture notes on network information theory,” January 2010, arXiv:1001.3404.
- [11] M. M. Wilde, *From Classical to Quantum Shannon Theory*, 2011, arXiv:1106.1445.
- [12] P. Sen, “Sequential decoding for some channels with classical input and quantum output,” 2011.
- [13] M. Hayashi and H. Nagaoka, “General formulas for capacity of classical-quantum channels,” *IEEE Transactions on Information Theory*, vol. 49, no. 7, pp. 1753–1768, 2003.
- [14] V. Giovannetti, S. Lloyd, and L. Maccone, “Achieving the Holevo bound via sequential measurements,” December 2010, arXiv:1012.0386.
- [15] O. Fawzi, P. Hayden, I. Savov, P. Sen, and M. M. Wilde, “Classical communication over a quantum interference channel,” February 2011, arXiv:1102.2624.
- [16] H. Sato, “The capacity of the Gaussian interference channel under strong interference (corresp.),” *IEEE Transactions on Information Theory*, vol. 27, no. 6, pp. 786–788, 1981.
- [17] M. H. M. Costa and A. El Gamal, “The capacity region of the discrete memoryless interference channel with strong interference,” *IEEE Transactions on Information Theory*, vol. 33, no. 5, pp. 710–711, 1987.
- [18] E. Sasoglu, “Successive cancellation for cyclic interference channels,” in *IEEE Information Theory Workshop, 2008. ITW’08*, 2008, pp. 36–40.
- [19] H. F. Chong, M. Motani, H. K. Garg, and H. El Gamal, “On the Han-Kobayashi region for the interference channel,” *IEEE Transactions on Information Theory*, vol. 54, no. 7, pp. 3188–3195, 2008.