

# Quantum Rate Distortion, Reverse Shannon Theorems, and Source-Channel Separation

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**Abstract**—We derive quantum counterparts of two key theorems of classical information theory, namely, the rate-distortion theorem and the source-channel separation theorem. The rate-distortion theorem gives the ultimate limits on lossy data compression, and the source-channel separation theorem implies that a two-stage protocol consisting of compression and channel coding is optimal for transmitting a memoryless source over a memoryless channel. In spite of their importance in the classical domain, there has been surprisingly little work in these areas for quantum information theory. In this paper, we prove that the quantum rate-distortion function is given in terms of the regularized entanglement of purification. We also determine a single-letter expression for the entanglement-assisted quantum rate-distortion function, and we prove that it serves as a lower bound on the unassisted quantum rate-distortion function. This implies that the unassisted quantum rate-distortion function is nonnegative and generally not equal to the coherent information between the source and distorted output (in spite of Barnum’s conjecture that the coherent information would be relevant here). Moreover, we prove several quantum source-channel separation theorems. The strongest of these are in the entanglement-assisted setting, in which we establish a necessary and sufficient condition for transmitting a memoryless source over a memoryless quantum channel up to a given distortion.

**Index Terms**—Quantum data compression, quantum rate distortion, quantum Shannon theory, reverse Shannon theorem, source-channel separation.

## I. INTRODUCTION

TWO pillars of classical information theory are Shannon’s data compression theorem and his channel capacity theorem [21], [49]. The former gives a fundamental limit to the compressibility of classical information, while the latter determines the ultimate limit on classical communication rates over

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a noisy classical channel. Modern communication systems exploit these ideas in order to make the best possible use of communication resources.

Data compression is possible due to statistical redundancy in the information emitted by sources, with some signals being emitted more frequently than others. Exploiting this redundancy suitably allows one to compress data without losing essential information. If the data which are recovered after the compression–decompression process is an exact replica of the original data, then the compression is said to be *lossless*. The simplest example of an information source is a memoryless one. Such a source can be characterized by a random variable  $U$  with probability distribution  $\{p_U(u)\}$  and each use of the source results in a letter  $u$  being emitted with probability  $p_U(u)$ . Shannon’s noiseless coding theorem states that the entropy  $H(U) \equiv -\sum_u p_U(u) \log_2 p_U(u)$  of such an information source is the minimum rate at which we can compress signals emitted by it [21], [49].

The requirement of a data compression scheme being lossless is often too stringent a condition, in particular for the case of multimedia data, i.e., audio, video, and still images or in scenarios where insufficient storage space is available. Typically, a substantial amount of data can be discarded before the information is sufficiently degraded to be noticeable. A data compression scheme is said to be *lossy* when the decompressed data are not required to be identical to the original one, but instead recovering a reasonably good approximation of the original data is considered to be good enough.

The theory of lossy data compression, which is also referred to as rate-distortion theory, was developed by Shannon [11], [21], [50]. This theory deals with the tradeoff between the rate of data compression and the allowed distortion. Shannon proved that, for a given memoryless information source and a distortion measure, there is a function  $R(D)$ , called the rate-distortion function, such that, if the maximum allowed distortion is  $D$ , then the best possible compression rate is given by  $R(D)$ . He established that this rate-distortion function is equal to the minimum of the mutual information  $I(U; \hat{U}) := H(U) + H(\hat{U}) - H(U, \hat{U})$  over all possible stochastic maps  $p_{\hat{U}|U}(\hat{u}|u)$  that meet the distortion requirement on average

$$R(D) = \min_{p_{\hat{U}|U}: \mathbb{E}\{d(U, \hat{U})\} \leq D} I(U; \hat{U}). \quad (1)$$

In the above,  $d(U, \hat{U})$  denotes a suitably chosen distortion measure between the random variable  $U$  characterizing the source and the random variable  $\hat{U}$  characterizing the output of the stochastic map.

Whenever the distortion  $D = 0$ , the aforementioned rate-distortion function is equal to the entropy of the source. If  $D > 0$ , then the rate-distortion function is less than the entropy, implying that fewer bits are needed to transmit the source if we allow for some distortion in its reconstruction.

Alongside these developments, Shannon also contributed the theory of reliable communication of classical data over classical channels [21], [49]. His noisy channel coding theorem gives an explicit expression for the capacity of a memoryless classical channel, i.e., the maximum rate of reliable communication through it. A memoryless channel  $\mathcal{N}$  is one for which there is no correlation in the noise acting on successive inputs, and it can be modeled by a stochastic map  $\mathcal{N} \equiv p_{Y|X}(y|x)$ . Shannon proved that the capacity of such a channel is given by

$$C(\mathcal{N}) = \max_{p_X(x)} I(X; Y).$$

Any scheme for error correction typically requires the use of redundancy in the transmitted data, so that the receiver can perfectly distinguish the received signals from one another in the limit of many uses of the channel.

Given all of the above results, we might wonder whether it is possible to transmit an information source  $U$  reliably over a noisy channel  $\mathcal{N}$ , such that the output of the information source is recoverable with an error probability that is asymptotically small in the limit of a large number of outputs of the information source and uses of the noisy channel. An immediate corollary of Shannon's noiseless and noisy channel coding theorems is that reliable transmission of the source is possible if the entropy of the source is smaller than the capacity of the channel

$$H(U) \leq C(\mathcal{N}). \quad (2)$$

The scheme to demonstrate sufficiency of (2) is for the sender to take the length  $n$  output of the information source, compress it down to  $nH(U)$  bits, and encode these  $nH(U)$  bits into a length  $n$  sequence for transmission over the channel. As long as  $H(U) \leq C(\mathcal{N})$ , Shannon's noisy channel coding theorem guarantees that it is possible to transmit the  $nH(U)$  bits over the channel reliably such that the receiver can decode them, and Shannon's noiseless coding theorem guarantees that the decoded  $nH(U)$  bits can be decompressed reliably as well in order to recover the original length  $n$  output of the information source (all of this is in the limit as  $n \rightarrow \infty$ ). Given that the condition in (2) is sufficient for reliable communication of the information source, is it also necessary? Shannon's source-channel separation theorem answers this question in the affirmative [21], [49].

The most important implication of the source-channel separation theorem is that we can consider the design of compression codes and channel codes separately—a two-stage encoding method is just as good as any other method, whenever the source and channel are memoryless. Thus, we should consider data compression and error correction as independent problems, and try to design the best compression scheme and the best error correction scheme. The source-channel separation theorem guarantees that this two-stage encoding and decoding with the best data compression and error correction codes will be optimal.

Now what if the entropy of the source is *greater* than the capacity of the channel? Our best hope in this scenario is to allow for some distortion in the output of the source such that the rate of compression is smaller than the entropy of the source. Recall that whenever  $D > 0$ , the rate-distortion function  $R(D)$  is less than the entropy  $H(U)$  of the source. In this case, we have a variation of the source-channel separation theorem which states that the condition  $R(D) \leq C(\mathcal{N})$  is both necessary and sufficient for the reliable transmission of an information source over a noisy channel, up to some amount of distortion  $D$ [21]. Thus, we can consider the problems of lossy data compression and channel coding separately, and the two-stage concatenation of the best lossy compression code with the best channel code is optimal.

Considering the importance of all of the aforementioned theorems for classical information theory, it is clear that theorems in this spirit would be just as important for quantum information theory. Note, however, that in the quantum domain, there are many different information processing tasks, depending on which type of information we are trying to transmit and which resources are available to assist the transmission. For example, we could transmit classical or quantum data over a quantum channel, and such a transmission might be assisted by entanglement shared between sender and receiver before communication begins.

There have been many important advances in the aforementioned directions (some of which are summarized in [57]). Schumacher proved the noiseless quantum coding theorem, demonstrating that the von Neumann entropy of a quantum information source is the ultimate limit to the compressibility of information emitted by it [45]. Hayashi has also considered many ways to compress quantum information, a summary of which is available in [30].

Quantum rate-distortion theory, which is the theory of lossy quantum data compression, was introduced by Barnum in 1998. He considered a symbol-wise entanglement fidelity as a distortion measure [4] and, with respect to it, defined the quantum rate-distortion function as the minimum rate of data compression, for any given distortion. He derived a lower bound on the quantum rate-distortion function, in terms of a well-known entropic quantity, namely the coherent information. The latter can be viewed as one quantum analog of mutual information, since it is known to characterize the quantum capacity of a channel [23], [38], [52], just as the mutual information characterizes the capacity of a classical channel. It is this analogy, and the fact that the classical rate-distortion function is given in terms of the mutual information, that led Barnum to consider the coherent information as a candidate for the rate-distortion function in the quantum realm. He also conjectured that this lower bound would be achievable.

Since Barnum's paper, there have been a few papers in which the problem of quantum rate distortion has either been addressed [20], [25], or mentioned in other contexts [31], [39], [40], [60]. However, not much progress has been made in proving or disproving his conjecture. In fact, in the absence of a matching upper bound, it is even unclear how good Barnum's bound is, given that the coherent information can be negative, as was pointed out in [20] and [25].

There are also a plethora of results on information transmission over quantum channels. Holevo [32], Schumacher, and Westmoreland [48] provided a characterization of the classical capacity of a quantum channel. Lloyd [38], Shor [52], and Devetak [23] proved that the coherent information of a quantum channel is an achievable rate for quantum communication over that channel, building on prior work of Nielsen and coworkers [5], [6], [46], [47] who showed that its regularization is an upper bound on the quantum capacity (note that the coherent information of a quantum channel is always nonnegative because it involves a maximization over all inputs to the channel). Bennett *et al.* proved that the mutual information of a quantum channel is equal to its entanglement-assisted classical capacity [10] (the capacity whenever the sender and receiver are given a large amount of shared entanglement before communication begins).

In [10], the authors also introduced the idea of a reverse Shannon theorem, in which a sender and receiver simulate a noisy channel with as few noiseless resources as possible (later papers rigorously proved several quantum reverse Shannon theorems [1], [8], [12]). Although such a task might initially seem unmotivated, they used a particular reverse Shannon theorem to establish a strong converse for the entanglement-assisted classical capacity.<sup>1</sup> Interestingly, the reverse Shannon theorems can also find application in rate-distortion theory [31], [39], [40], [60], and as such, they are relevant for our purposes here.

In this paper, we prove several important quantum rate-distortion theorems and quantum source-channel separation theorems. Our first result in quantum rate distortion is a complete characterization of the rate-distortion function in an entanglement-assisted setting.<sup>2</sup> This result really only makes sense in the communication paradigm (and *not* in a storage setting), where we give a sender and receiver shared entanglement before communication begins, in addition to the uses of the noiseless qubit channel. The idea here is for a sender to exploit the shared entanglement and a minimal amount of classical or quantum communication in order for the receiver to recover the output of the quantum information source up to some distortion. Our main result is a single-letter formula for the entanglement-assisted rate-distortion function, expressed in terms of a minimization of the input–output mutual information over all quantum operations that meet the distortion constraint. This result implies that the computation of the entanglement-assisted rate-distortion function for *any* quantum information source is a tractable convex optimization program. It is often the case in quantum Shannon theory that the entanglement-assisted formulas end up being formally analogous to Shannon’s classical formulas [10], [28], and our result here is no exception to this trend.

We next consider perhaps the most natural setting for quantum rate distortion in which a compressor tries to compress a quantum information source so that a decompressor can

<sup>1</sup>A strong converse demonstrates that the error probability asymptotically approaches one if the rate of communication is larger than capacity. This is in contrast to a weak converse, which only demonstrates that the error probability is bounded away from zero under the same conditions.

<sup>2</sup>One might consider these entanglement-assisted rate-distortion results to be part of the “quantum reverse Shannon theorem folklore,” but [8] does not specifically discuss this topic.

recover it up to some distortion  $D$  (this setting is the same as Barnum’s in [4]). This setting is most natural whenever sufficient quantum storage is not available, but we can equivalently phrase it in a communication paradigm, where a sender has access to many uses of a noiseless qubit channel and would like to minimize the use of this resource while transmitting a quantum information source up to some distortion. We prove that the quantum rate-distortion function is given in terms of a regularized entanglement of purification [55] in this case. In spite of our characterization being an intractable, regularized formula, our result at the very least shows that the quantum rate-distortion function is always nonnegative, demonstrating that Barnum’s conjecture from [4] does not hold since his proposed rate-distortion function can become negative. Furthermore, we prove that the entanglement-assisted quantum rate-distortion function is a single-letter lower bound on the unassisted quantum rate-distortion function (one might suspect that this should hold because additional resources such as shared entanglement should only be able to improve compression rates). This bound implies that the coherent information between the source and distorted output is not relevant for unassisted quantum rate distortion, in spite of Barnum’s conjecture that it would be.

We finally prove three source-channel separation theorems that apply to the transmission of a classical source over a quantum channel, the transmission of a quantum source over a quantum channel, and the transmission of a quantum source over an entanglement-assisted quantum channel, respectively. The first two source-channel separation theorems are single letter, in the sense that they do not involve any regularized quantities, whenever the Holevo capacity or the coherent information of the channel is additive, respectively. The third theorem is single letter in all cases because the entanglement-assisted quantum capacity is given by a single-letter expression for all quantum channels [2], [10]. We also prove a related set of source-channel separation theorems that allow for some distortion in the reconstruction of the output of the information source. From these theorems, we infer that it is best to search for the best quantum data compression protocols [3], [9], [13], [16], [42], [43], the best quantum error-correcting codes [18], [19], [37], [41], [44], [51], and the best entanglement-assisted quantum error-correcting codes [17], [33], [36], [58] independently of each other whenever the source and channel are memoryless. The theorems then guarantee that combining these protocols in a two-stage encoding and decoding is optimal.

We structure this paper as follows. We first overview relevant notation and definitions in the next section. Section III introduces the information processing task relevant for quantum rate distortion and then presents all of our quantum rate-distortion results in detail. Section IV presents our various quantum source-channel separation theorems for memoryless sources and channels. Finally, we conclude in Section V and discuss important open questions.

## II. NOTATION AND DEFINITIONS

Let  $\mathcal{H}$  denote a finite-dimensional Hilbert space and let  $\mathcal{D}(\mathcal{H})$  denote the set of density matrices or *states* (i.e., positive oper-

ators of unit trace) acting on  $\mathcal{H}$ . Let  $\rho_A \in \mathcal{D}(\mathcal{H}_A)$  denote the state characterizing a memoryless quantum information source, the subscript  $A$  being used to denote the underlying quantum system. We refer to it as the *source state*. Let  $|\psi_{RA}^\rho\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$  denote its *purification*, i.e.,

$$\psi_{RA}^\rho = |\psi_{RA}^\rho\rangle\langle\psi_{RA}^\rho|$$

is a pure state density matrix of a larger composite system  $RA$ , such that its restriction to the system  $A$  is given by  $\rho_A$ , i.e.,  $\rho_A := \text{Tr}_R \psi_{RA}^\rho$ , with  $\text{Tr}_R$  denoting the partial trace over the Hilbert space  $\mathcal{H}_R$  of a purifying reference system  $R$ . The pure state  $|\psi_{RA}^\rho\rangle$  is entangled if  $\rho$  is a mixed state. The von Neumann entropy of  $\rho_A$ , and hence of the source, is defined as

$$H(A)_\rho \equiv -\text{Tr} \{\rho \log \rho\}. \quad (3)$$

The quantum mutual information of a bipartite state  $\omega_{AB}$  is defined as

$$I(A; B)_\omega \equiv H(A)_\omega + H(B)_\omega - H(AB)_\omega.$$

The coherent information  $I(A)B)_\sigma$  of a bipartite state  $\sigma_{AB}$  is defined as follows:

$$I(A)B)_\sigma := H(B)_\sigma - H(AB)_\sigma. \quad (4)$$

In quantum information theory, the most general mathematical description of any allowed physical operation is given by a completely positive trace-preserving (CPTP) map, which is a map between states. We let  $\text{id}_A$  denote the trivial (or identity) CPTP map which keeps the state of a quantum system  $A$  unchanged, and we let  $\mathcal{N} \equiv \mathcal{N}^{A \rightarrow B}$  denote the CPTP map

$$\mathcal{N}^{A \rightarrow B} : \mathcal{D}(\mathcal{H}_A) \mapsto \mathcal{D}(\mathcal{H}_B).$$

The entanglement of purification of a bipartite state  $\omega_{AB}$  is a measure of correlations [55], having an operational interpretation as the entanglement cost of creating  $\omega_{AB}$  asymptotically from ebits, while consuming a negligible amount of classical communication. It is equivalent to the following expression:

$$E_p(\omega_{AB}) \equiv \min_{\mathcal{N}_E} H((\text{id}_B \otimes \mathcal{N}_E)(\mu_{BE}(\omega)))$$

where  $\mu_{BE}(\omega) = \text{Tr}_A \{\phi_{ABE}^\omega\}$ ,  $\phi_{ABE}^\omega$  is some purification of  $\omega_{AB}$ , and the minimization is over all CPTP maps  $\mathcal{N}_E$  acting on the system  $E$ . (The original definition in [55] is different from the above, but one can check that the definition given here is equivalent to the one given there.)

In this paper, we make use of resource inequalities (see, e.g., [26]), to express information-processing tasks as interconversions between resources. Let  $[c \rightarrow c]$  denote one forward use of a noiseless classical bit channel,  $[q \rightarrow q]$  one forward use of a noiseless qubit channel, and  $[qq]$  one ebit of shared entanglement (a Bell state). A simple example of a resource inequality is entanglement distribution

$$[q \rightarrow q] \geq [qq]$$

meaning that Alice can consume one noiseless qubit channel in order to generate one ebit between her and Bob. Teleportation is a more interesting way in which all three resources interact [7]

$$2[c \rightarrow c] + [qq] \geq [q \rightarrow q].$$

The aforementioned resource inequalities are finite and exact, but we can also express quantum Shannon theoretic protocols as resource inequalities. For example, the resource inequality for the protocol achieving the entanglement-assisted classical capacity of a quantum channel is as follows:

$$\langle \mathcal{N} \rangle + H(A) [qq] \geq I(A; B) [c \rightarrow c].$$

The meaning of the aforementioned resource inequality is that there exists a protocol exploiting  $n$  uses of a memoryless quantum channel  $\mathcal{N}$  and  $nH(A)$  ebits in order to transmit  $nI(A; B)$  classical bits from sender to receiver. The resource inequality becomes exact in the asymptotic limit  $n \rightarrow \infty$  because it is possible to show that the error probability of decoding these classical bits correctly approaches zero as  $n \rightarrow \infty$  [10].

### III. QUANTUM RATE DISTORTION

#### A. Information Processing Task

The objective of any quantum rate-distortion protocol is to compress a quantum information source such that the decompressor can reconstruct the original state up to some distortion. Like Barnum [4], we consider the following distortion measure  $d(\rho, \mathcal{N})$  for a state  $\rho \in \mathcal{D}(\mathcal{H}_A)$  with purification  $|\psi_{RA}^\rho\rangle$  and a quantum operation  $\mathcal{N} \equiv \mathcal{N}^{A \rightarrow B}$ :

$$d(\rho, \mathcal{N}) = 1 - F_e(\rho, \mathcal{N}) \quad (5)$$

where  $F_e$  is the entanglement fidelity of the map  $\mathcal{N}$

$$F_e(\rho, \mathcal{N}) \equiv \langle \psi_{RA}^\rho | (\text{id}_R \otimes \mathcal{N}^{A \rightarrow B})(\psi_{RA}^\rho) | \psi_{RA}^\rho \rangle. \quad (6)$$

The entanglement fidelity is not only a natural distortion measure, but it also possesses several analytical properties which prove useful in our analysis.

The state  $\rho^n := (\rho)^{\otimes n} \in \mathcal{D}(\mathcal{H}_A^{\otimes n})$  characterizes  $n$  successive outputs of a memoryless quantum information source. A source coding (or compression–decompression) scheme of rate  $R$  is defined by a block code, which consists of two quantum operations—the encoding and decoding maps. The encoding  $\mathcal{E}_n$  is a map from  $n$  copies of the source space to a subspace  $\mathcal{H}_{Q^n} \subset \mathcal{H}_A^{\otimes n}$  of dimension  $2^{nR}$

$$\mathcal{E}_n : \mathcal{D}(\mathcal{H}_A^{\otimes n}) \rightarrow \mathcal{D}(\tilde{\mathcal{H}}_{Q^n})$$

and the decoding  $\mathcal{D}_n$  is a map from the compressed subspace to the output Hilbert space  $\mathcal{H}_A^{\otimes n}$

$$\mathcal{D}_n : \mathcal{D}(\tilde{\mathcal{H}}_{Q^n}) \rightarrow \mathcal{D}(\mathcal{H}_A^{\otimes n}).$$

The average distortion resulting from this compression-decompression scheme is defined as [4]

$$\bar{d}(\rho, \mathcal{D}_n \circ \mathcal{E}_n) \equiv \sum_{i=1}^n \frac{1}{n} d(\rho, \mathcal{F}_n^{(i)})$$

where  $\mathcal{F}_n^{(i)}$  is the ‘‘marginal operation’’ on the  $i$ th copy of the source space induced by the overall operation  $\mathcal{F}_n \equiv \mathcal{D}_n \circ \mathcal{E}_n$ , and is defined as

$$\mathcal{F}_n^{(i)}(\rho) \equiv \text{Tr}_{A_1, A_2, \dots, A_{i-1}, A_{i+1}, \dots, A_n} [\mathcal{F}_n(\rho^{\otimes n})]. \quad (7)$$

The quantum operations  $\mathcal{D}_n$  and  $\mathcal{E}_n$  define an  $(n, R)$  quantum rate-distortion code.

For any  $R, D \geq 0$ , the pair  $(R, D)$  is said to be an *achievable* rate-distortion pair if there exists a sequence of  $(n, R)$  quantum rate-distortion codes  $(\mathcal{E}_n, \mathcal{D}_n)$  such that

$$\lim_{n \rightarrow \infty} \bar{d}(\rho, \mathcal{D}_n \circ \mathcal{E}_n) \leq D. \quad (8)$$

The quantum rate-distortion function is then defined as

$$R^q(D) = \inf \{ R : (R, D) \text{ is achievable} \}.$$

In the communication model, if the sender and receiver have unlimited prior shared entanglement at their disposal, then the corresponding quantum rate-distortion function is denoted as  $R_{\text{eac}}^q(D)$  or  $R_{\text{eaq}}^q(D)$ , depending on whether the noiseless channel between the sender and the receiver is classical or quantum. Fig. 1 depicts the most general protocols for unassisted and assisted quantum rate-distortion coding.

### B. Reverse Shannon Theorems and Quantum Rate-Distortion Coding

Before we begin with our main results, we first prove Lemma 1 in the following. This lemma is similar in spirit to Lemma 26 of [39] and Theorem 19 of [60], and like them, it shows that to generate a rate-distortion code, it suffices to simulate the action of a noisy channel on a source state such that the resulting output state meets the desired distortion criterion. Unlike them, however, it is specifically tailored to the entanglement fidelity distortion measure.

*Lemma 1:* Fix  $\varepsilon > 0$  and  $0 \leq D < 1$ . Consider a state  $\rho_A$  with purification  $|\psi_{RA}^\rho\rangle$  and a quantum channel  $\mathcal{N} \equiv \mathcal{N}^{A \rightarrow B}$  for which  $d(\rho, \mathcal{N}) \leq D$ . Let

$$\omega_{RB} := (\text{id} \otimes \mathcal{N}) \psi_{RA}^\rho.$$

Furthermore, let  $\{\mathcal{F}_n\}_n$  denote a sequence of quantum operations such that for  $n$  large enough

$$\|\sigma_{R^n B^n} - \omega_{RB}^{\otimes n}\|_1 \leq \varepsilon \quad (9)$$

for some  $\varepsilon \geq 0$  where

$$\sigma_{R^n B^n} := (\text{id}_{R^n} \otimes \mathcal{F}_n) ((\psi_{RA}^\rho)^{\otimes n}).$$

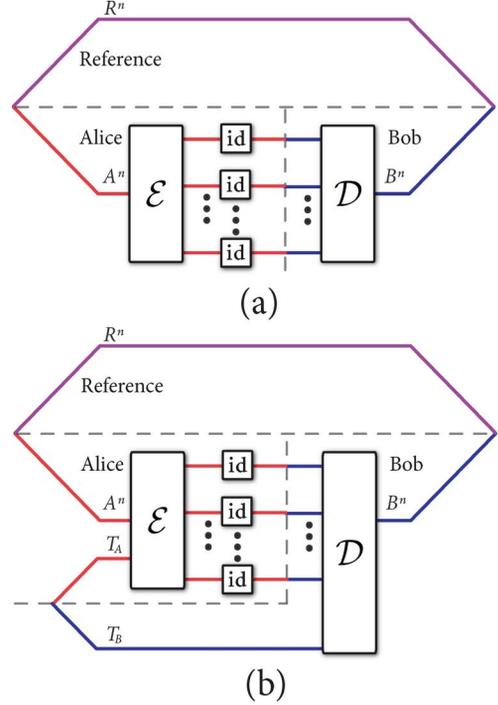


Fig. 1. Most general protocols for (a) unassisted and (b) assisted quantum rate-distortion coding. In (a), Alice acts on the tensor power output of the quantum information source with a compression encoding  $\mathcal{E}$ . She sends the compressed qubits over noiseless quantum channels (labeled by ‘‘id’’) to Bob, who then performs a decompression map  $\mathcal{D}$  to recover the quantum data that Alice sent. In (b), the task is similar, though this time we assume that Alice and Bob share entanglement before communication begins.

Then, for  $n$  large enough, the average distortion under the quantum operation  $\mathcal{F}_n$  satisfies the bound

$$\bar{d}(\rho, \mathcal{F}_n) \leq D + \varepsilon$$

*Proof:* Expressing  $R^n = R_1 R_2 \cdots R_n$ , and  $B^n = B_1 B_2 \cdots B_n$ , we have for any  $1 \leq i \leq n$

$$\sigma_{R_i B_i} = (\text{id}_R \otimes \mathcal{F}_n^{(i)}) (\psi_{RA}^\rho). \quad (10)$$

By monotonicity of the trace distance under partial trace, we have that

$$\|\sigma_{R_i B_i} - \omega_{RB}\|_1 \leq \|\sigma_{R^n B^n} - \omega_{RB}^{\otimes n}\|_1. \quad (11)$$

Hence, the average distortion under the quantum operation  $\mathcal{F}_n$  is given by

$$\begin{aligned} \bar{d}(\rho, \mathcal{F}_n) &= \frac{1}{n} \sum_{i=1}^n \left( 1 - F_e(\rho, \mathcal{F}_n^{(i)}) \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left( 1 - \langle \psi_{RA}^\rho | \sigma_{R_i B_i} | \psi_{RA}^\rho \rangle \right). \end{aligned} \quad (12)$$

Recall the following inequality from [15]:

$$\text{Tr}(P(A - B)) \geq \text{Tr}(A - B)_- \quad (13)$$

where  $0 \leq P \leq I$  is any positive operator and  $(A - B)_-$  denotes the negative spectral part of the operator  $(A - B)$ . We then have the following inequalities:

$$\begin{aligned} & \langle \psi_{RA}^\rho | \sigma_{R_i B_i} | \psi_{RA}^\rho \rangle \\ &= \langle \psi_{RA}^\rho | \omega_{RB} | \psi_{RA}^\rho \rangle + \text{Tr}(\psi_{RA}^\rho (\sigma_{R_i B_i} - \omega_{RB})) \\ &\geq F_e(\rho, \mathcal{N}) + \text{Tr}(\sigma_{R_i B_i} - \omega_{RB})_- \end{aligned} \quad (14)$$

where the inequality follows from (13) and the definition of entanglement fidelity

$$\langle \psi_{RA}^\rho | \omega_{RB} | \psi_{RA}^\rho \rangle = F_e(\rho, \mathcal{N}).$$

Hence, from (12), (14) and (11), we have

$$\begin{aligned} & \bar{d}(\rho, \mathcal{F}_n) \\ &\leq \frac{1}{n} \sum_{i=1}^n [1 - F_e(\rho, \mathcal{N}) - \text{Tr}(\sigma_{R_i B_i} - \omega_{R_i B_i})_-] \\ &\leq \frac{1}{n} \sum_{i=1}^n [1 - F_e(\rho, \mathcal{N}) + \|\sigma_{R_i B_i} - \omega_{R_i B_i}\|_1] \\ &\leq d(\rho, \mathcal{N}) + \|\sigma_{R^n B^n} - \omega_{R^n B^n}\|_1 \\ &\leq D + \varepsilon \end{aligned} \quad (15)$$

which concludes the proof of the lemma.  $\blacksquare$

The aforementioned lemma illustrates a fundamental connection between quantum reverse Shannon theorems and quantum rate-distortion protocols. In particular, if a reverse Shannon theorem is available in a given context, then it immediately leads to a rate-distortion protocol. This is done simply by choosing the simulated channel to be the one which, when acting on the source state, yields an output state which meets the distortion criterion for the desired rate-distortion task. This is our approach in all of the quantum rate-distortion theorems that follow, and it was also the approach in [25], [39], and [60].

There is, however, one caveat with the aforementioned approach. The reverse Shannon theorems often require extra correlated resources such as shared randomness or shared entanglement [1], [8], [10], [12], and the demands of a reverse Shannon theorem are much more stringent than those of a rate-distortion protocol. A reverse Shannon theorem requires the simulation of a channel to be asymptotically exact, whereas a rate-distortion protocol only demands that a source be reconstructed up to some average distortion constraint. The differences in these goals can impact resulting rates if sufficient correlated resources are not available [22].

In the entanglement-assisted setting considered in the next section, the assumption is that an unlimited supply of entanglement is available, and thus, the entanglement-assisted quantum reverse Shannon theorem suffices for producing a good entanglement-assisted rate-distortion protocol. In the unassisted setting, no correlation is available, and exploiting the unassisted reverse Shannon theorem leads to rates that are possibly larger

than necessary for the task of quantum rate distortion. Nevertheless, we still employ this approach and discuss the ramifications further in the forthcoming sections.

### C. Entanglement-Assisted Rate-Distortion Coding

1) *Rate Distortion With Noiseless Classical Communication:* The quantum rate-distortion function  $R_{\text{eac}}^q(D)$  for entanglement-assisted lossy source coding with noiseless classical communication is given by the following theorem.

*Theorem 2:* For a memoryless quantum information source defined by the density matrix  $\rho_{A'}$ , with a purification  $|\psi_{AA'}^\rho\rangle$ , and any given distortion  $0 \leq D < 1$ , the quantum rate-distortion function for entanglement-assisted lossy source coding with noiseless classical communication is given by

$$R_{\text{eac}}^q(D) = \min_{\mathcal{N}: d(\rho, \mathcal{N}) \leq D} I(A; B)_\omega \quad (16)$$

where  $\mathcal{N} \equiv \mathcal{N}^{A' \rightarrow B}$  denotes a CPTP map

$$\omega_{AB} \equiv (\text{id}_A \otimes \mathcal{N}^{A' \rightarrow B})(\psi_{AA'}^\rho)$$

and  $I(A; B)_\omega$  denotes the quantum mutual information.

*Proof:* We first prove the converse (optimality). Consider the most general protocol for entanglement-assisted lossy source coding that acts on many copies ( $\rho^{\otimes n}$ ) of the state  $\rho \in \mathcal{D}(\mathcal{H}_A)$  [depicted in Fig. 1(b)]. We take a purification of  $\rho$  as  $|\psi_{RA}^\rho\rangle$ . Let  $\Phi_{T_A T_B}$  denote an entangled state, with the system  $T_A$  being with Alice and the system  $T_B$  being with Bob. Alice then acts on the state  $\rho^{\otimes n}$  and her share  $T_A$  of the entangled state with a compression map  $\mathcal{E}_n \equiv \mathcal{E}^{A^n T_A \rightarrow W}$ , where  $W$  is a classical system of size  $\approx 2^{nr}$ , with  $r$  being the rate of compression [in Fig. 1(b),  $W$  corresponds to the outputs of the noiseless quantum channels]. Then, Bob acts on both the classical system  $W$  that he receives and his share  $T_B$  of the entangled state with the decoding map  $\mathcal{D}_n \equiv \mathcal{D}^{W T_B \rightarrow B^n}$ . The final state should be such that it is distorted by at most  $D$  according to the average distortion criterion in the limit  $n \rightarrow \infty$  (8). With these steps in mind, consider the following chain of inequalities:

$$\begin{aligned} nr &\geq H(W) \\ &\geq H(W|T_B) \\ &\geq H(W|T_B) - H(W|R^n T_B) \\ &= I(W; R^n | T_B) \\ &= I(W; R^n | T_B) + I(R^n; T_B) \\ &= I(W T_B; R^n) \\ &\geq I(B^n; R^n). \end{aligned}$$

The first inequality follows because the entropy  $nr$  of the uniform distribution is the largest that the entropy  $H(W)$  can be. The second inequality follows because conditioning cannot increase entropy. The third inequality follows because  $H(W|R^n T_B) \geq 0$  from the assumption that  $W$  is classical.

The first equality follows from the definition of mutual information, and the second equality follows from the fact that  $R^n$  and  $T_B$  are in a product state. The third equality is the chain rule for quantum mutual information. The final inequality is from quantum data processing. Continuing, we have

$$\begin{aligned}
 &\geq \sum_{i=1}^n I(B_i; R_i) \\
 &\geq \sum_{i=1}^n R_{\text{eac}}^q \left( d \left( \rho, \mathcal{F}_n^{(i)} \right) \right) \\
 &= n \sum_{i=1}^n \frac{1}{n} R_{\text{eac}}^q \left( d \left( \rho, \mathcal{F}_n^{(i)} \right) \right) \\
 &\geq n R_{\text{eac}}^q \left( \sum_{i=1}^n \frac{1}{n} d \left( \rho, \mathcal{F}_n^{(i)} \right) \right) \\
 &\geq n R_{\text{eac}}^q (D). \tag{17}
 \end{aligned}$$

In the above,  $\mathcal{F}_n^{(i)}$  is the marginal operation on the  $i$ th copy of the source space induced by the overall operation  $\mathcal{F}_n \equiv \mathcal{D}_n \circ \mathcal{E}_n$ , and is given by (7). The first inequality follows from superadditivity of quantum mutual information (see Lemma 15 in the Appendix). The second inequality follows from the fact that the map  $\mathcal{D}_i \circ \mathcal{E}_i$  has distortion  $d(\rho, \mathcal{D}_i \circ \mathcal{E}_i)$  and the information rate-distortion function is the minimum of the mutual information over all maps with this distortion. The last two inequalities follow from convexity of the quantum rate-distortion function  $R_{\text{eac}}^q(D)$  (see Lemma 14 in the Appendix), from the assumption that the average distortion of the protocol is no larger than the amount allowed

$$\sum_{i=1}^n \frac{1}{n} d(\rho, \mathcal{D}_i \circ \mathcal{E}_i) \leq D$$

and from the fact that  $R_{\text{eac}}^q(D)$  is nonincreasing as a function of  $D$  (see Lemma 14 in the Appendix).

The direct part of Theorem 2 follows from the quantum reverse Shannon theorem, which states that it is possible to simulate (asymptotically perfectly) the action of a quantum channel  $\mathcal{N}$  on an arbitrary state  $\rho$ , by exploiting noiseless classical communication and prior shared entanglement between a sender and receiver [1], [8], [10], [12]. The resource inequality for this protocol is

$$I(A; B)_\omega [c \rightarrow c] + H(B)_\omega [qq] \geq \langle \mathcal{N} : \rho \rangle \tag{18}$$

where the entropies are with respect to a state of the following form:

$$|\omega_{ABE}\rangle \equiv U_{\mathcal{N}}^{A' \rightarrow BE} |\psi_{AA'}^\rho\rangle$$

$|\psi_{AA'}^\rho\rangle$  is a purification of  $\rho$ ,  $U_{\mathcal{N}}^{A' \rightarrow BE}$  is an isometric extension of the channel  $\mathcal{N}^{A' \rightarrow B}$ . Our protocol simply exploits this theorem. More specifically, for a given distortion  $D$ , we take  $\mathcal{N}$  to be the CPTP map which achieves the minimum in the expression (16) of  $R_{\text{eac}}^q(D)$ . Then, we exploit classical communication at the rate given in the resource inequality (18) to simulate the action of the channel  $\mathcal{N}$  on the source state  $\rho$ . For any arbitrarily small  $\varepsilon > 0$  and  $n$  large enough, the protocol for the

quantum reverse Shannon theorem simulates the action of the channel up to the constant  $\varepsilon$  (in the sense of (9)). This allows us to invoke Lemma 1 to show that the resulting average distortion is no larger than  $D + \varepsilon$ . ■

The main reason that we can use the quantum reverse Shannon theorem as a ‘‘black box’’ for the purpose of quantum rate distortion is from our assumption of unlimited shared entanglement. It is likely that this protocol uses much more entanglement than necessary for the purpose of entanglement-assisted quantum rate-distortion coding with classical channels, and it should be worthwhile to study the tradeoff between classical communication and entanglement consumption in more detail, as previous authors have done in the context of channel coding [34], [35], [53], [59]. Such a study might lead to a better protocol for entanglement-assisted rate-distortion coding and might further illuminate better protocols for other quantum rate-distortion tasks.

We think that our protocol exploits more entanglement than necessary from considering what is known in the classical case regarding reverse Shannon theorems and rate-distortion coding [10], [21], [22]. First, as reviewed in (1), the classical mutual information minimized over all stochastic maps that meet the distortion criterion is equal to Shannon’s classical rate-distortion function [21]. Bennett *et al.* have shown that the classical mutual information is also equal to the minimum rate needed to simulate a classical channel whenever free common randomness is available [10]. Thus, a simple strategy for achieving the task of rate distortion is for the parties to choose the stochastic map that minimizes the rate-distortion function and simulate it with the classical reverse Shannon theorem. But this strategy uses far more classical bits than necessary whenever sufficient common randomness is not available [22]. Meanwhile, we already know that the mutual information is achievable without any common randomness if the goal is rate distortion [21].

2) *Rate Distortion With Noiseless Quantum Communication:* The quantum rate-distortion function  $R_{\text{eac}}^q(D)$  for entanglement-assisted lossy source coding with noiseless quantum communication is given by the following theorem.

*Theorem 3:* For a memoryless quantum information source defined by the density matrix  $\rho_{A'}$ , with a purification  $|\psi_{AA'}^\rho\rangle$ , and any given distortion  $0 \leq D < 1$ , the quantum rate-distortion function for entanglement-assisted lossy source coding with noiseless quantum communication is given by

$$R_{\text{eac}}^q(D) = \frac{1}{2} \left[ \min_{\mathcal{N}: d(\rho, \mathcal{N}) \leq D} I(A; B)_\omega \right] \tag{19}$$

where  $\mathcal{N} \equiv \mathcal{N}^{A' \rightarrow B}$  denotes a CPTP map

$$\omega_{AB} \equiv (\text{id}_A \otimes \mathcal{N}^{A' \rightarrow B})(\psi_{AA'}^\rho) \tag{20}$$

and  $I(A; B)_\omega$  denotes the quantum mutual information.

*Proof:* We first prove the converse (optimality). The setup is similar to that in the converse proof of Theorem 2, with the exception that  $W$  is now a quantum system of dimension  $2^{nr}$  and

we let  $E$  denote the environment of the compressor. Consider the following chain of inequalities:

$$\begin{aligned}
2nr &\geq 2H(W) \\
&= H(W) + H(R^n T_B E) \\
&\geq H(W) + H(R^n T_B E) - H(W R^n T_B E) \\
&= I(W; R^n T_B E) \\
&\geq I(W; R^n T_B) \\
&= I(W T_B; R^n) + I(W; T_B) - I(R^n; T_B) \\
&= I(W T_B; R^n) + I(W; T_B) \\
&\geq I(W T_B; R^n) \\
&\geq I(B^n; R^n). \tag{21}
\end{aligned}$$

The first inequality is because the entropy  $nr$  of the uniform distribution is the largest that the entropy  $H(W)$  can be. The first equality follows from the fact that the state on systems  $W R^n T_B E$  is pure. The second inequality follows by subtracting the positive quantity  $H(W R^n T_B E)$ . The second equality is from the definition of quantum mutual information. The third inequality is from quantum data processing (tracing over system  $E$ ). The third equality is a useful identity for quantum mutual information. The fourth equality follows from  $I(R^n; T_B) = 0$  since  $R^n$  and  $T_B$  are in a product state. The second-to-last inequality is from  $I(W; T_B) \geq 0$ , and the final inequality is from the quantum data processing inequality. The rest of the proof proceeds as in (17).

The direct part follows from a variant of the quantum reverse Shannon theorem known as the fully quantum reverse Shannon theorem (FQRS) [1], [24]. This theorem states that it is possible to simulate (asymptotically perfectly) the action of a channel  $\mathcal{N}$  on an arbitrary state  $\rho$ , by exploiting noiseless quantum communication and prior shared entanglement between a sender and receiver. It has the following resource inequality:

$$\frac{1}{2}I(A; B)_\omega [q \rightarrow q] + \frac{1}{2}I(B; E)_\omega [qq] \geq \langle \mathcal{N} : \rho \rangle \tag{22}$$

where the entropies are with respect to a state of the following form:

$$|\omega_{ABE}\rangle \equiv U_{\mathcal{N}}^{A' \rightarrow BE} |\psi_{AA'}^\rho\rangle \tag{23}$$

$|\psi_{AA'}^\rho\rangle$  is a purification of  $\rho$ , and  $U_{\mathcal{N}}^{A' \rightarrow BE}$  is an isometric extension of the channel  $\mathcal{N}^{A' \rightarrow B}$ . Our protocol exploits this theorem as follows. For a given distortion  $D$ , take  $\mathcal{N}$  to be the map which realizes the minimum in the expression (19) of  $R_{\text{eq}}^q(D)$ . Then, we exploit quantum communication at the rate given in the resource inequality (22) to simulate the action of the channel  $\mathcal{N}$  on the source state  $\rho$ . For any arbitrarily small  $\varepsilon > 0$  and  $n$  large enough, the protocol for the FQRS simulates the action of the channel up to the constant  $\varepsilon$  (in the sense of (9)). This allows us to invoke Lemma 1 to show that the resulting average distortion is no larger than  $D + \varepsilon$ . ■

We could have determined that the form of the entanglement-assisted quantum rate-distortion function  $R_{\text{eq}}^q(D)$  in Theorem

3 follows easily from Theorem 2 by combining with teleportation. Though, the aforementioned proof serves an important alternate purpose. A careful inspection of it reveals that the steps detailed in (21) for bounding the quantum communication rate still hold even if the system  $T_B$  is trivial (in the case where there is no shared entanglement between the sender and receiver before communication begins). Thus, we obtain as a corollary that the entanglement-assisted quantum rate-distortion function is a single-letter lower bound on the unassisted quantum rate-distortion function. This makes sense operationally as well because the additional resource of shared entanglement should only be able to improve a rate-distortion protocol.

*Corollary 4:* The entanglement-assisted quantum rate-distortion function  $R_{\text{eq}}^q(D)$  in Theorem 3 bounds the unassisted quantum rate-distortion function  $R^q(D)$  from below

$$R^q(D) \geq R_{\text{eq}}^q(D).$$

The above corollary firmly asserts that the coherent information  $I(A)B$  of the state in (20) is not relevant for quantum rate distortion, in spite of Barnum's conjecture that it would play a role [4]. That is, one might think that there should be some simple fix of Barnum's conjecture, say, by conjecturing that the quantum rate-distortion function would instead be  $\max\{0, I(A)B\}$ . The aforementioned lower bound asserts that this cannot be the case because half the mutual information is never smaller than the coherent information

$$\frac{1}{2}I(A; B) \geq \frac{1}{2}I(A; B) - \frac{1}{2}I(A; E) = I(A)B.$$

#### D. Unassisted Quantum Rate-Distortion Coding

The quantum rate-distortion function  $R^q(D)$  for unassisted lossy source coding is given by the following theorem.

*Theorem 5:* For a memoryless quantum information source defined by the density matrix  $\rho_A$ , and any given distortion  $0 \leq D < 1$ , the quantum rate-distortion function is given by

$$R^q(D) = \lim_{k \rightarrow \infty} \frac{1}{k} \min_{\substack{\mathcal{N}^{(k)} \\ d(\rho^{\otimes k}, \mathcal{N}^{(k)}) \leq D}} [E_p(\rho^{\otimes k}, \mathcal{N}^{(k)})] \tag{24}$$

where  $\mathcal{N}^{(k)} : \mathcal{D}(\mathcal{H}_A^{\otimes k}) \rightarrow \mathcal{D}(\mathcal{H}_B^{\otimes k})$  is a CPTP map, and

$$E_p(\rho, \mathcal{N}) \equiv E_p(\omega_{RB}) \tag{25}$$

denotes the entanglement of purification, with

$$\omega_{RB} \equiv (\text{id}_R \otimes \mathcal{N}^{A \rightarrow B})(\psi_{RA}^\rho). \tag{26}$$

Like its classical counterpart, lossy data compression includes lossless compression as a special case. If the distortion  $D$  is set equal to zero in (24), then the state  $\omega_{RB}$  becomes identical to the state  $\psi_{RA}^\rho$ . Equivalently, the quantum operation  $\mathcal{N}$  is given by the identity map  $\text{id}_A$ . Since the entanglement of purification is additive for tensor power states [55]

$$E_p((\psi_{RA}^\rho)^{\otimes n}) = nE_p(\psi_{RA}^\rho) = nS(\rho_A)$$

we infer that, for  $D = 0$ ,  $R^q(D)$  reduces to the von Neumann entropy of the source, which is known to be the optimal rate for lossless quantum data compression [45].

To prove the achievability part of Theorem 5, we can simply exploit Schumacher compression [45] (which is a special type of reverse Shannon theorem). Alice feeds each output  $A$  of the source into a CPTP map  $\mathcal{N}$  that saturates the bound in (24) (for now, we do not consider the limit and set  $k = 1$ ). This leads to a state of the form in (26), to which Alice can then apply Schumacher compression. This protocol is equivalent to the following resource inequality:

$$H(B)_\omega [q \rightarrow q] \geq \langle \mathcal{N} : \rho \rangle. \quad (27)$$

We note that this is a simple form of an unassisted quantum reverse Shannon theorem.

Now, a subtle detail of the simulation idea is that we are interested in simulating the channel  $\mathcal{N}^{A \rightarrow B}$  from Alice to Bob, and Alice can actually simulate an isometric extension  $U_{\mathcal{N}}^{A \rightarrow BE}$  of the channel where Alice receives the system  $E$  and just traces over it.

Though, instead of simulating  $U_{\mathcal{N}}^{A \rightarrow BE}$ , we could consider Alice to simulate the isometry  $U_{\mathcal{N}}^{A \rightarrow BE_B E_A}$  locally, Schumacher compressing the subsystems  $B$  and  $E_B$  so that Bob can recover them, while the subsystem  $E_A$  remains with Alice. This leads to the following protocol for unassisted simulation:

$$H(BE_B)_\omega [q \rightarrow q] \geq \langle \mathcal{N} : \rho \rangle.$$

The best protocol for unassisted channel simulation is therefore the one with the minimum rate of quantum communication, the minimum being taken over all possible isometries  $V : E \rightarrow E_A E_B$ . This rate can only be less than the rate of quantum communication required for the original naive protocol in (27) since the latter is a special case in the minimization. This is the form of the unassisted quantum reverse Shannon theorem given in [8] and is related to a protocol considered by Hayashi [29].

One could then execute the aforementioned protocol by blocking  $k$  of the states together and by having the distortion channel be of the form  $\mathcal{N}^{(k)} : A^k \rightarrow B^{(k)}$ , acting on each block of  $k$  states. By letting  $k$  become large, such a protocol leads to the following rate for unassisted communication:

$$Q_{\min}(\rho, \mathcal{N}) = \lim_{k \rightarrow \infty} \frac{1}{k} \min_{V: E^{(k)} \rightarrow E_A E_B} H(B^{(k)} E_B). \quad (28)$$

The aforementioned quantity is equal to the entanglement of purification of the state  $(\text{id}_R \otimes \mathcal{N}^{A^k \rightarrow B^{(k)}})((\psi_{RA}^\rho)^{\otimes k})$  [8], [29]

$$\begin{aligned} & \lim_{k \rightarrow \infty} \frac{1}{k} \min_{V: E^{(k)} \rightarrow E_A E_B} H(B^{(k)} E_B) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \min_{\Lambda^{E^{(k)} \rightarrow E_B}} H(\Lambda^{E^{(k)} \rightarrow E_B} ((U_{\mathcal{N}}^{A^k \rightarrow B^{(k)} E^{(k)}} (\rho_A^{\otimes k}))) \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} E_p((\text{id}_{R^k} \otimes \mathcal{N}^{A^k \rightarrow B^{(k)}})((\psi_{RA}^\rho)^{\otimes k})). \end{aligned}$$

We are now in a position to prove Theorem 5.

*Proof of Theorem 5:* Fix the map  $\mathcal{N}$  such that the minimization on the RHS of (24) is achieved. The quantum reverse Shannon theorem (in this case, Schumacher compression) states that it is possible to simulate such a channel  $\mathcal{N}$  acting on  $\rho$  with the amount of quantum communication equal to  $E_p(\omega_{RB})$ . Since the protocol simulates the channel up to some arbitrarily small positive  $\varepsilon$ , the distortion is no larger than  $D + \varepsilon$  by invoking Lemma 1. This establishes that  $R^q(D) \geq E_p(\omega_{RB})$ . We can have a regularization as earlier to obtain the expression in the statement of the theorem.

The converse part of the theorem can be proved as follows. Fig. 1(a) depicts the most general protocol for unassisted quantum rate-distortion coding. Let  $E_1$  denote the environment of the encoder, and let  $E_2$  denote the environment of the decoder, while  $W$  again denotes the outputs of the noiseless quantum channels labeled by ‘‘id.’’ For any rate-distortion code  $(\mathcal{E}^{(n)}, \mathcal{D}^{(n)})$  of rate  $r$  satisfying  $\bar{d}(\rho, \mathcal{D}^{(n)} \circ \mathcal{E}^{(n)}) \leq D$ , we have

$$\begin{aligned} nr &\geq H(W) \\ &= H(E_2 B^n)_\omega \\ &\geq \min_{\Lambda_{E_1 E_2}} H((\text{id}_{B^n} \otimes \Lambda_{E_1 E_2})(\omega_{B^n E_1 E_2})) \\ &= E_p\left((\text{id}_{R^n} \otimes (\mathcal{D}^{(n)} \circ \mathcal{E}^{(n)}))(\psi_{RA}^\rho)^{\otimes n}\right) \\ &\geq \min_{\mathcal{N}^{(n)}: \bar{d}(\rho^{\otimes n}, \mathcal{N}^{(n)}) \leq D} E_p\left((\text{id}_{R^n} \otimes \mathcal{N}^{(n)})(\psi_{RA}^\rho)^{\otimes n}\right). \end{aligned} \quad (29)$$

The first inequality follows because the entropy of the maximally mixed state is larger than the entropy of any state on system  $W$ . The first equality follows because the isometric extension of the decoder maps  $W$  isometrically to the systems  $E_2$  and  $B^n$ . The second inequality follows because the entropy minimized over all CPTP maps on systems  $E_1$  and  $E_2$  can only be smaller than the entropy on  $E_2 B^n$  (the identity map on  $E_2$  and partial trace of  $E_1$  is a CPTP map included in the minimization). The second equality follows from the definition of entanglement of purification. The third inequality follows by minimizing the entanglement of purification over all maps that satisfy the distortion criterion (recall that we assume our protocol satisfies this distortion criterion). ■

Our characterization of the unassisted quantum rate-distortion task is unfortunately up to a regularization. It is likely that this regularized formula is blurring a better quantum rate-distortion formula, as has sometimes been the case in quantum Shannon theory [61]. This is due in part to our exploitation of the unassisted reverse Shannon theorem for the task of quantum rate distortion, and the fact that the goal of a reverse Shannon theorem is stronger than that of a rate-distortion protocol, while no correlated resources are available in this particular setting (see the previous discussion after Theorem 2). It would be ideal to demonstrate that the regularization is not necessary, but it is not clear yet how to do so without a better way to realize unassisted quantum rate distortion. Nevertheless, the aforementioned theorem at the very least disproves Barnum’s conjecture because we have demonstrated that the quantum rate-distortion function

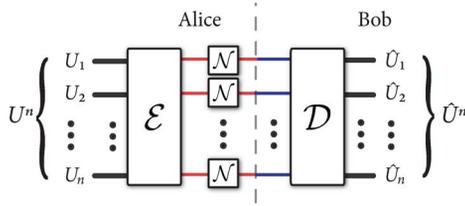


Fig. 2. Most general protocol for transmitting a classical information source over a memoryless quantum channel.

is always positive (due to the fact that entanglement of purification is positive [55]), whereas Barnum's rate-distortion function can become negative.<sup>3</sup> Furthermore, Corollary 4 provides a good single-letter, nonnegative lower bound on the unassisted quantum rate-distortion function, which is never smaller than Barnum's bound in terms of the coherent information.

#### IV. SOURCE-CHANNEL SEPARATION Theorems

This last section of our paper consists of five important quantum source-channel separation theorems. The first two theorems apply whenever a sender wishes to transmit a memoryless classical source over a memoryless quantum channel, whereas the third applies when the information source to be transmitted is a quantum source. The second theorem deals with the situation in which some distortion is allowed in the transmission. All these three theorems are expressed in terms of single-letter formulas whenever the corresponding capacity formulas are single letter.

The last two theorems correspond to the cases in which a quantum source is sent over an entanglement-assisted quantum channel, with and without distortion. The formulas in these are always single letter, demonstrating that it is again the entanglement-assisted formulas which are in formal analogy with Shannon's classical formulas.

##### A. Shannon's Source-Channel Separation Theorem for Quantum Channels

Shannon's original source-channel separation theorem applies to the transmission of a classical information source over a classical channel. Despite the importance of this theorem, it does not take into account that the carriers of information are essentially quantum-mechanical. So our first theorem is a restatement of Shannon's source-channel separation theorem for the case in which a classical information source is to be reliably transmitted over a quantum channel.

Fig. 2 depicts the scenario to which this first source-channel separation theorem applies. The most general protocol for sending the output of a classical information source over a quantum channel consists of three steps: encoding, transmission, and decoding. The sender first takes the outputs  $U^n$  of the

<sup>3</sup>To see that Barnum's proposed distortion function can become negative, consider the case of a maximally mixed qubit source, whose purification is the maximally entangled Bell state. Suppose that we allow the distortion to be as large as 3/4. Then, a particular map satisfying the distortion criterion is the completely depolarizing map because it produces a tensor product of maximally mixed qubits, whose entanglement fidelity with the maximally entangled state is equal to 1/4. The coherent information of a tensor product of maximally mixed qubits is equal to its minimum value of  $-1$ .

classical information source and encodes them with some CPTP encoding map  $\mathcal{E}^{U^n \rightarrow A^n}$ , where the systems  $A^n$  are the inputs to many uses of a noisy quantum channel  $\mathcal{N}^{A \rightarrow B}$ . The sender then transmits the systems  $A^n$  over the quantum channels, and the receiver obtains the outputs  $B^n$ . The receiver finally performs some CPTP decoding map  $\mathcal{D}^{B^n \rightarrow \hat{U}^n}$  to recover the random variables  $\hat{U}^n$  (note that this decoding is effectively a positive operator-valued measure because the output systems are classical). If the scheme is any good for transmitting the source, then the following condition holds for any given  $\varepsilon > 0$ , for sufficiently large  $n$ :

$$\Pr \left\{ \hat{U}^n \neq U^n \right\} \leq \varepsilon. \quad (30)$$

*Theorem 6:* The following condition is necessary and sufficient for transmitting the output of a memoryless classical information source, characterized by a random variable  $U$ , over a memoryless quantum channel  $\mathcal{N} \equiv \mathcal{N}^{A \rightarrow B}$ , with additive Holevo capacity:

$$H(U) \leq \chi^*(\mathcal{N}) \quad (31)$$

where

$$\begin{aligned} \chi^*(\mathcal{N}) &\equiv \max_{\rho^{XA}} I(X; B)_\rho \\ \rho^{XB} &\equiv \sum_x p_X(x) |x\rangle \langle x|^X \otimes \mathcal{N}^{A \rightarrow B}(\rho_x^A). \end{aligned}$$

*Proof:* Sufficiency of (31) is a direct consequence of Shannon compression and Holevo–Schumacher–Westmoreland (HSW) coding. The sender first compresses the information source down to a set of size  $\approx 2^{nH(U)}$ . The sender then employs an HSW code to transmit any message in the compressed set over  $n$  uses of the quantum channel. Reliability of the scheme follows from the assumption that  $H(U) \leq \chi^*(\mathcal{N})$ , the HSW coding theorem, and Shannon compression.

Necessity of (31) follows from reasoning similar to that in the proof of the classical source-channel separation theorem [21]. Fix  $\varepsilon > 0$ . We begin by assuming that there exists a good scheme that meets the criterion in (30). Consider the following chain of inequalities:

$$\begin{aligned} nH(U) &= H(U^n) \\ &= I(U^n; \hat{U}^n) + H(U^n | \hat{U}^n) \\ &\leq I(U^n; \hat{U}^n) + 1 + \Pr\{\hat{U}^n \neq U^n\} n \log |U| \\ &\leq I(U^n; B^n) + 1 + \varepsilon n \log |U| \\ &\leq \chi^*(\mathcal{N}^{\otimes n}) + 1 + \varepsilon n \log |U| \\ &= n\chi^*(\mathcal{N}) + 1 + \varepsilon n \log |U|. \end{aligned} \quad (32)$$

The first equality follows from the assumption that the classical information source is memoryless. The second equality is a simple identity. The first inequality follows from applying Fano's inequality. The second inequality follows from the quantum data processing inequality and the assumption that (30) holds. The third inequality follows because  $I(U^n; B^n)$  must be smaller than the maximum of this quantity over all

classical-quantum states that can serve as an input to the tensor power channel  $\mathcal{N}^{\otimes n}$ . The final equality follows from the assumption that the Holevo capacity is additive for the particular channel  $\mathcal{N}$ . Thus, any protocol that reliably transmits the information source  $U$  should satisfy the following inequality:

$$H(U) \leq \chi^*(\mathcal{N}) + (1/n + \varepsilon \log |U|)$$

which converges to (31) as  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ . ■

*Remark 7:* If the Holevo capacity is not additive for the channel, then the best statement of the source-channel separation theorem is in terms of the regularized quantity

$$H(U) \leq \chi_{\text{reg}}^*(\mathcal{N})$$

where

$$\chi_{\text{reg}}^*(\mathcal{N}) \equiv \lim_{n \rightarrow \infty} \frac{1}{n} \chi^*(\mathcal{N}^{\otimes n})$$

but it is unclear how useful such a statement is because we cannot compute such a regularized quantity. (The aforementioned statement follows by applying all of the inequalities in the proof of Theorem 6 except the last one.)

What if the condition  $H(U) > \chi^*(\mathcal{N})$  holds instead? We can prove a variant of the earlier source-channel separation theorem that allows for the information source to be reconstructed at the receiving end up to some distortion  $D$ . We obtain the following theorem.

*Theorem 8:* The following condition is necessary and sufficient for transmitting the output of a memoryless classical information source over a quantum channel with additive Holevo capacity (up to some distortion  $D$ ):

$$R(D) \leq \chi^*(\mathcal{N}) \quad (33)$$

where  $R(D)$  is defined in (1).

*Proof:* Sufficiency of (33) follows from the rate-distortion protocol and the HSW coding theorem. Specifically, the sender compresses the information source down to a set of size  $2^{nR(D)}$  and then uses an HSW code to transmit any element of this set. The reconstructed sequence  $\hat{U}^n$  at the receiving end obeys the distortion constraint  $\mathbb{E}\{d(U, \hat{U})\} \leq D$ , with  $d(U, \hat{U})$  denoting a suitably defined distortion measure.

Necessity of (33) follows from the fact that

$$nR(D) \leq I(U^n; \hat{U}^n) \quad (34)$$

and by applying the last four steps in the chain of inequalities in (32). A proof of (34) is available in [21, (10.61–10.71)]. ■

### B. Quantum Source-Channel Separation Theorem

We now prove a source-channel separation theorem which is perhaps more interesting for quantum computing/communication applications. Suppose that a sender would like to transmit a quantum information source faithfully over a quantum channel, such that the receiver perfectly recovers the transmitted quantum source in the limit of many copies of the

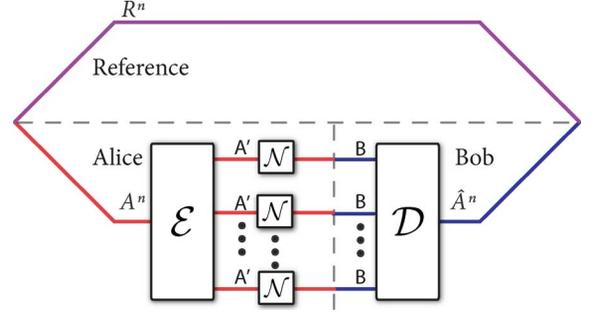


Fig. 3. Most general protocol for transmitting a quantum information source over a memoryless quantum channel.

source and uses of the channel. Fig. 3 depicts the scenario to which our second source-channel separation theorem applies.

As earlier, we characterize a memoryless quantum information source by a density matrix  $\rho_A \in \mathcal{D}(\mathcal{H}_A)$ , and let  $|\psi_{RA}^\rho\rangle \in \mathcal{H}_R \otimes \mathcal{H}_A$  denote its purification. The entropy of the source  $H(A)_\rho$  is given by (3). Let  $\mathcal{N}^{A' \rightarrow B}$  denote a memoryless quantum channel. Suppose Alice has access to multiple uses of the source, and she and Bob are allowed multiple uses of the quantum channel.

Since Alice needs to act on many copies of the state  $\rho$ , we instead suppose that she is acting on the  $A$  systems of the tensor power state  $|\psi_{RA}^\rho\rangle^{\otimes n}$ . The most general protocol is one in which Alice performs some CPTP encoding map  $\mathcal{E}_n \equiv \mathcal{E}^{A^n \rightarrow A'^n}$  on the  $A$  systems of the state  $|\psi_{RA}^\rho\rangle^{\otimes n}$ , producing some output systems  $A'^n$  which can serve as input to many uses of the quantum channel  $\mathcal{N}^{A' \rightarrow B}$ . Alice then transmits the  $A'^n$  systems over the channels, leading to some output systems  $B^n$  for Bob. Bob then acts on these systems with some decoding map  $\mathcal{D}_n \equiv \mathcal{D}^{B^n \rightarrow \hat{A}^n}$ . If the protocol is any good for transmitting the quantum information source, then the following condition should hold for any  $\varepsilon > 0$  and sufficiently large  $n$ :

$$\left\| \left( |\psi_{RA}^\rho\rangle^{\otimes n} - \mathcal{D}_n(\mathcal{N}^{\otimes n}(\mathcal{E}_n(|\psi_{RA}^\rho\rangle^{\otimes n}))) \right) \right\|_1 \leq \varepsilon. \quad (35)$$

The relation between trace distance and entanglement fidelity [57] implies that

$$F_e(\rho^{\otimes n}, \Lambda_n) \geq 1 - \varepsilon \quad (36)$$

where  $\Lambda_n$  is the composite map  $\Lambda_n \equiv \mathcal{D}_n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}_n$ .

We can now state our first variant of a quantum source-channel separation theorem.

*Theorem 9:* The following condition is necessary and sufficient for transmitting the output of a memoryless quantum information source, characterized by a density matrix  $\rho_A$ , over a quantum channel  $\mathcal{N} \equiv \mathcal{N}^{A' \rightarrow B}$  with additive coherent information:

$$H(A)_\rho \leq Q(\mathcal{N}) \quad (37)$$

where  $H(A)_\rho$  is the entropy of the quantum information source, and  $Q(\mathcal{N})$  is the coherent information of the channel  $\mathcal{N}$

$$Q(\mathcal{N}) \equiv \max_{|\phi_{AA'}\rangle} I(A)B)_\sigma$$

$$\sigma_{AB} \equiv \mathcal{N}^{A' \rightarrow B}(\phi_{AA'}).$$

*Proof:* Sufficiency of (37) follows from Schumacher compression and the direct part of the quantum capacity theorem [23], [38], [52]. Specifically, the sender compresses the source down to a space of dimension  $\approx 2^{nH(R)}$  with the Schumacher compression protocol. She then encodes this subspace with a quantum error correction code for the channel  $\mathcal{N}$ . The condition in (37) guarantees that we can apply the direct part of the quantum capacity theorem, and combined with achievability of Schumacher compression, the receiver can recover the quantum information source with asymptotically small error in the limit of many copies of the source and many uses of the quantum channel.

Fix  $\varepsilon > 0$  and note that  $H(A)_\rho = H(R)_\psi$  since  $\psi_{RA}^\rho$  is a pure state. Then, the necessity of (37) follows from the chain of inequalities given as follows. Note that the subscripts denoting the states have been omitted for simplicity

$$\begin{aligned}
nH(A) &= nH(R) \\
&= H(R^n) \\
&\leq I(R^n)B^n + 2 + 4(1 - F_e) \log |R^n| \\
&\leq I(R^n)B^n M + 2 + 4\varepsilon n \log |R| \\
&= \sum_m p(m) I(R^n)B^n_{\rho_m} + 2 + 4\varepsilon n \log |R| \\
&\leq Q(\mathcal{N}^{\otimes n}) + 2 + 4\varepsilon n \log |R| \\
&= nQ(\mathcal{N}) + 2 + 4\varepsilon n \log |R|. \tag{38}
\end{aligned}$$

The first equality follows from the assumption that the initial state  $|\psi_{RA}^\rho\rangle^{\otimes n}$  is a tensor power state. The first inequality follows from [6, (7.34)], a fundamental relation between the input entropy, the coherent information of a channel, and the entanglement fidelity of any quantum error correction code.

Now, the encoding that Alice employs may in general be some CPTP encoding map (and not an isometry). However, Alice can simulate any such CPTP map by first performing an isometry and then a von Neumann measurement on the system not fed into the channel (the environment of the simulated CPTP). Let  $M$  denote the classical system resulting from measuring the environment of the simulated CPTP map. We can write the state after the channel acts as a classical-quantum state of the following form:

$$\sum_m p(m) |m\rangle \langle m|_M \otimes \rho_{R^n B^n}^m.$$

Then, the second inequality follows from quantum data processing inequality and (36). The second equality follows because

$$\begin{aligned}
I(R^n)B^n M &= I(R^n)B^n |M\rangle \\
&= \sum_m p(m) I(R^n)B^n_{\rho_m}
\end{aligned}$$

whenever the conditioning system is classical [57]. The third inequality follows because the channel's coherent information is never smaller than any individual  $I(R^n)B^n_{\rho_m}$  (and thus never smaller than the average). The final inequality follows from the assumption that the channel has additive coherent information (this holds for degradable quantum channels [27] and is suspected to hold for two-Pauli channels [54]). Thus, any protocol

that reliably transmits the quantum information source should satisfy the following inequality:

$$H(R) \leq Q(\mathcal{N}) + (2/n + 4\varepsilon \log |R|)$$

which converges to (37) as  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ .  $\blacksquare$

*Remark 10:* A similar comment as in Remark 7 holds whenever it is not known that the channel has additive coherent information.

### C. Entanglement-Assisted Quantum Source-Channel Separation Theorem

Our final source-channel separation theorem applies to the scenario where Alice and Bob have unlimited prior shared entanglement. The statement of this theorem is that the entropy of the quantum information source being less than the entanglement-assisted quantum capacity of the channel [10], [26], [57] is both a necessary and sufficient condition for the faithful transmission of the source over an entanglement-assisted quantum channel. This theorem is the most powerful of any of the above because the formulas involved are all single letter, for any memoryless source and channel.

Fig. 4 depicts the scenario to which this last theorem applies. The situation is nearly identical to that of the previous section, with the exception that Alice and Bob have unlimited prior shared entanglement. Alice begins by performing some CPTP encoding map  $\mathcal{E}_n \equiv \mathcal{E}^{A^n T_A \rightarrow A'^n}$  on the systems  $A^n$  from the quantum information source and on her share  $T_A$  of the entanglement, producing some output systems  $A'^n$  which can serve as input to many uses of a quantum channel  $\mathcal{N}^{A' \rightarrow B}$ . Alice then transmits the  $A'^n$  systems over the channels, leading to some output systems  $B^n$  for Bob. Bob then acts on these systems and his share  $T_B$  of the entanglement with some decoding map  $\mathcal{D}_n \equiv \mathcal{D}^{B^n T_B \rightarrow \hat{A}^n}$ . If the protocol is any good for transmitting the quantum information source, then the following condition should hold for any  $\varepsilon > 0$  and sufficiently large  $n$ :

$$\left\| \left( \psi_{RA}^\rho \right)^{\otimes n} - \mathcal{D}_n \left( \mathcal{N}^{\otimes n} \left( \mathcal{E}_n \left( \left( \psi_{RA}^\rho \right)^{\otimes n} \otimes \Phi^{T_A T_B} \right) \right) \right) \right\|_1 \leq \varepsilon \tag{39}$$

where  $\Phi^{T_A T_B}$  is the entangled state that they share before communication begins (it does not necessarily need to be maximally entangled). This leads to our final source-channel separation theorem.

*Theorem 11:* The following condition is necessary and sufficient for transmitting the output of a memoryless quantum information source, characterized by a density matrix  $\rho_A$ , over any entanglement-assisted quantum channel  $\mathcal{N} \equiv \mathcal{N}^{A' \rightarrow B}$ :

$$H(A)_\rho \leq \frac{1}{2} I(\mathcal{N}) \tag{40}$$

where  $H(A)_\rho$  is the entropy of the quantum information source and

$$\begin{aligned}
I(\mathcal{N}) &\equiv \max_{|\varphi_{AA'}\rangle} I(A; B)_\sigma \\
\sigma_{AB} &\equiv \mathcal{N}^{A' \rightarrow B}(\varphi_{AA'}).
\end{aligned}$$

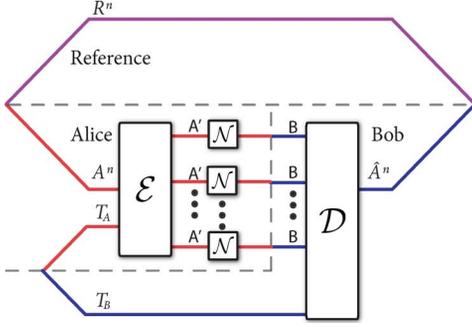


Fig. 4. Most general protocol for transmitting a quantum information source over a memoryless, entanglement-assisted quantum channel.

*Proof:* Sufficiency of (40) follows from reasoning similar to that in the proof of Theorem 9. We just exploit Schumacher compression and the entanglement-assisted quantum capacity theorem [10], [26], [57].

Fix  $\varepsilon > 0$  and note that  $H(A)_\rho = H(R)_\psi$  since  $\psi_{RA}^\rho$  is a pure state. Then, necessity of (40) follows from the following chain of inequalities. Once again, the subscripts denoting the states have been omitted for simplicity

$$\begin{aligned}
 2nH(R) &= 2H(R^n) \\
 &\leq H(R^n) + I(R^n; B^n T_B) \\
 &\quad + 2 + 4(1 - F_e) \log |R^n| \\
 &\leq I(R^n; B^n T_B) + 2 + 4n\varepsilon \log |R| \\
 &= I(R^n T_B; B^n) + I(R^n; T_B) - I(T_B; B^n) \\
 &\quad + 2 + 4n\varepsilon \log |R| \\
 &= I(R^n T_B; B^n) - I(T_B; B^n) + 2 + 4n\varepsilon \log |R| \\
 &\leq I(R^n T_B M; B^n) + 2 + 4n\varepsilon \log |R| \\
 &\leq \max_{\rho_{XAA'^n}} I(A X; B^n) + 2 + 4n\varepsilon \log |R| \\
 &= I(\mathcal{N}^{\otimes n}) + 2 + 4n\varepsilon \log |R| \\
 &= nI(\mathcal{N}) + 2 + 4n\varepsilon \log |R|. \tag{41}
 \end{aligned}$$

The first inequality follows by applying the same reasoning as the first inequality in (38). The second inequality follows by applying  $H(R^n) + I(R^n; B^n T_B) = I(R^n; B^n T_B)$  and the fact that  $1 - F_e \leq \varepsilon$  for a protocol satisfying (39). The second inequality follows from a useful identity for quantum mutual information. The third equality follows from the assumption that systems  $R^n$  and  $T_B$  begin in a product state. The third inequality follows because  $I(T_B; B^n) \geq 0$ . The fourth inequality follows from the reasoning, similar to that used in the proof of Theorem 9, that Alice simulates an isometry and measures the environment (also exploiting the quantum data processing inequality). The next inequality follows because the state on  $R^n T_B M B^n$  is a state of the form

$$\sum_x p_X(x) |x\rangle \langle x|_X \otimes \mathcal{N}^{A'^n \rightarrow B^n}(\rho_{AA'^n}^x)$$

where we identify  $R^n T_B$  with  $A$ , and  $M$  with  $X$ . Thus, the information quantity  $I(R^n T_B M; B^n)$  can never be larger than the maximum over all such states of that form. The second-to-

last equality was proved in [57] and [59]. The final equality follows from additivity of the channel's quantum mutual information [2], [10], [57]. Thus, any entanglement-assisted protocol that reliably transmits the quantum information source should satisfy the following inequality:

$$H(A)_\rho \leq \frac{1}{2}I(\mathcal{N}) + (1/n + 2\varepsilon \log |R|)$$

which converges to (40) as  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ .  $\blacksquare$

What if the condition  $H(A)_\rho > \frac{1}{2}I(\mathcal{N})$  holds instead? We can prove a variant of the aforementioned source-channel separation theorem that allows for the information source to be reconstructed at the receiving end up to some distortion  $D$ . We obtain the following theorem.

*Theorem 12:* The following condition is necessary and sufficient for transmitting the output of a memoryless quantum information source over an entanglement-assisted quantum channel (up to some distortion  $D$ ):

$$R_{\text{eaq}}^q(D) \leq \frac{1}{2}I(\mathcal{N}) \tag{42}$$

where  $R_{\text{eaq}}^q(D)$  is defined (19).

*Proof:* Sufficiency of (42) follows from the entanglement-assisted rate-distortion protocol from Theorem 3 and the entanglement-assisted quantum capacity theorem [10], [26]. That is, the sender compresses the information source down to a space of size  $2^{nR_{\text{eaq}}^q(D)}$  and then uses an entanglement-assisted quantum code to transmit any state in this subspace. The reconstructed state at the receiving end obeys the distortion constraint.

Necessity of (42) follows from the fact that

$$nR_{\text{eaq}}^q(D) \leq \frac{1}{2}I(R^n; \hat{A}^n) \tag{43}$$

by applying the quantum data processing inequality to get  $I(R^n; \hat{A}^n) \leq I(R^n; B^n T_B)$ , and finally by applying the last seven steps in the chain of inequalities in (41). A proof of (43) is available in (17) of the proof of Theorem 2.  $\blacksquare$

## V. CONCLUSION

We have proved several quantum rate-distortion theorems and quantum source-channel separation theorems. All of our quantum rate-distortion protocols employ the quantum reverse Shannon theorems [1], [8], [10], [12], [24]. This strategy works out well whenever unlimited entanglement is available, but it clearly leads to undesirable regularized formulas in the unassisted setting. Our quantum source-channel separation theorems demonstrate in many cases that a two-stage compression-channel-coding strategy works best for memoryless sources and for quantum channels with additive capacity measures. Again, our most satisfying result is in the entanglement-assisted setting, where the pleasing result is that the entanglement-assisted rate-distortion function being less than the entanglement-assisted quantum capacity is both necessary and sufficient for transmission of a source over a channel up to some distortion.

The most important open question going forward from here is to determine better protocols for quantum rate distortion that do not rely on the reverse Shannon theorems. The differing goals of a reverse Shannon theorem and a rate-distortion protocol are what lead to complications with regularization in Theorem 5.

Another productive avenue could be to explore scenarios where the unassisted quantum source-channel separation theorem does not apply. In the classical case, it is known that certain sources and channels without a memoryless structure can violate the source-channel separation theorem [56], and similar ideas would possibly demonstrate a violation for the quantum case. In the quantum case, it very well could be that certain memoryless sources and channels could violate source-channel separation, but we would need a better understanding of quantum capacity in the general case in order to determine definitively whether this could be so.

Other interesting questions are as follows: does the entanglement-assisted quantum source-channel separation theorem apply if sender and receiver are given unlimited access to a quantum feedback channel, given what we already know about quantum feedback [14]? Can anything learned from source-channel separation for classical broadcast or wiretap channels be applied to figure out a more general characterization for quantum channels that are not degradable?

#### APPENDIX SUPPORTING LEMMAS

*Lemma 13:* For a fixed state  $\rho$ , the quantum mutual information is convex in the channel operation

$$I(A; B)_\omega \leq \sum_x p(x) I(A; B)_{\omega_x}$$

where

$$\begin{aligned} \omega_{AB} &:= (\text{id} \otimes \mathcal{N}^{A' \rightarrow B})(\psi_{AA'}^\rho) \\ \omega_{AB}^x &:= (\text{id} \otimes \mathcal{N}_x^{A' \rightarrow B})(\psi_{AA'}^\rho) \\ \mathcal{N} &\equiv \sum_x p(x) \mathcal{N}_x. \end{aligned} \quad (44)$$

*Proof:* It is possible to show that

$$\begin{aligned} I(A; B)_\omega &= H(\rho) + H(\mathcal{N}(\rho)) - H((I \otimes \mathcal{N})(\psi)) \\ I(A; B)_{\omega_x} &= H(\rho) + H(\mathcal{N}_x(\rho)) - H((I \otimes \mathcal{N}_x)(\psi)) \end{aligned}$$

and the desired inequality becomes

$$\begin{aligned} &H(\rho) + H(\mathcal{N}(\rho)) - H((I \otimes \mathcal{N})(\psi)) \\ &\leq \sum_x p(x) [H(\rho) + H(\mathcal{N}_x(\rho)) - H((I \otimes \mathcal{N}_x)(\psi))]. \end{aligned}$$

This inequality is equivalent to

$$\begin{aligned} &H(\mathcal{N}(\rho)) - H((I \otimes \mathcal{N})(\psi)) \\ &\leq \sum_x p(x) [H(\mathcal{N}_x(\rho)) - H((I \otimes \mathcal{N}_x)(\psi))] \end{aligned}$$

which in turn is equivalent to convexity of coherent information, or equivalently, the quantum data processing inequality for coherent information

$$I(A)B \leq I(A)BX.$$

■

*Lemma 14:* The quantum rate-distortion function  $R_{\text{eac}}^q(D)$  is nonincreasing and convex

$$\begin{aligned} D_1 < D_2 &\Rightarrow R_{\text{eac}}^q(D_1) \geq R_{\text{eac}}^q(D_2) \\ R_{\text{eac}}^q(\lambda D_1 + (1 - \lambda) D_2) &\leq \lambda R_{\text{eac}}^q(D_1) + (1 - \lambda) R_{\text{eac}}^q(D_2) \end{aligned}$$

where  $0 \leq \lambda \leq 1$ .

*Proof:* The proof is similar to Barnum's [4], which in turn is similar to the one from [21].  $R_{\text{eac}}^q(D)$  is nonincreasing because the domain of minimization becomes larger after increasing  $D$ , which implies that the rate-distortion function can only become smaller. Let  $(R_1, D_1)$  and  $(R_2, D_2)$  be two points on the information rate-distortion curve and let  $\mathcal{E}_1$  and  $\mathcal{E}_2$  be the respective operations that achieve the minimum in the definition of  $R_{\text{eac}}^q$ , respectively. Consider the map  $\mathcal{E}_\lambda \equiv \lambda \mathcal{E}_1 + (1 - \lambda) \mathcal{E}_2$ . Under the assumption of a distortion function that is linear in the operation (such as the entanglement fidelity), it follows that the distortion caused by  $\mathcal{E}_\lambda$  is  $D_\lambda = \lambda D_1 + (1 - \lambda) D_2$ . We also have that  $R_{\text{eac}}^q(D_\lambda)$  is the minimum over all operations that have distortion  $D_\lambda$  so that  $R_{\text{eac}}^q(D_\lambda) \leq I(A; B)_\omega$  where  $\omega^{AB} \equiv \mathcal{E}_\lambda^{A' \rightarrow B}(\psi^{AA'})$ . Finally, we have that the mutual information is convex in the operation (see Lemma 13) so that  $I(A; B)_\omega \leq \lambda R_{\text{eac}}^q(D_1) + (1 - \lambda) R_{\text{eac}}^q(D_2)$ . ■

*Lemma 15 (Superadditivity of Mutual Information):* The mutual information is superadditive in the sense that

$$I(R_1 R_2; B_1 B_2) \geq I(R_1; B_1) + I(R_2; B_2)$$

where the entropies are with respect to the following state:

$$\theta_{R_1 R_2 B_1 B_2} \equiv \mathcal{N}^{A_1 A_2 \rightarrow B_1 B_2}(\phi_{R_1 A_1} \otimes \varphi_{R_2 A_2})$$

with  $\mathcal{N}^{A_1 A_2 \rightarrow B_1 B_2}$  some noisy channel, and  $\phi_{R_1 A_1}$  and  $\varphi_{R_2 A_2}$  being pure, bipartite states.

*Proof:* The inequality is equivalent to

$$\begin{aligned} &H(R_1 R_2) + I(R_1 R_2)_{B_1 B_2} \\ &\geq H(R_1) + I(R_1)_{B_1} + H(R_2) + I(R_2)_{B_2}. \end{aligned}$$

Observing that  $H(R_1 R_2) = H(R_1) + H(R_2)$  because the state on  $R_1$  and  $R_2$  is product, the inequality is equivalent to

$$I(R_1 R_2)_{B_1 B_2} \geq I(R_1)_{B_1} + I(R_2)_{B_2}$$

which is in turn equivalent to

$$I(R_1 B_1; R_2 B_2) \geq I(B_1; B_2).$$

This last inequality follows from the quantum data processing inequality. ■

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