Classical codes for quantum broadcast channels

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***Abstract*—We discuss two techniques for transmitting classical information over quantum broadcast channels. The first technique is a quantum generalization of the superposition coding scheme for the classical broadcast channel. We use a quantum simultaneous nonunique decoder and obtain a simpler proof of the rate region recently published by Yard *et al.* in independent work. Our second result is a quantum Marton coding scheme, which gives the best known achievable rate region for quantum broadcast channels. Both results exploit recent advances in quantum simultaneous decoding developed in the context of quantum interference channels.**

I. Introduction

How can a broadcast station communicate separate messages to two receivers using a single antenna? Two well known strategies [1] for transmitting information over broadcast channels are superposition coding [2], [3] and Marton over-binning using correlated auxiliary random variables [4]. In this paper, we prove that these strategies can be adapted to the quantum setting by constructing random codebooks and matching decoding measurements that have asymptotically vanishing error in the limit of many uses of the channel.

Sending classical data over a quantum channel is one of the fundamental problems of quantum information theory [5]. Single-letter formulas are known for classical-quantum point-to-point channels [6], [7] and multiple access channels [8]. Classical-quantum channels are a useful abstraction for studying general quantum channels and correspond to the transmitters being restricted to classical encodings. Codes for classical-quantum channels (c-q channels), when augmented with an extra optimization over the possible input states, directly generalize to codes for quantum channels. Furthermore, it is known that classical encoding (coherent-state encoding using classical Gaussian codebooks) is sufficient to achieve the capacity of the lossy bosonic channel, which is a realistic model for optical communication links [9].

Previous work on quantum broadcast channels includes [10], [11], [12]. Ref. [10] considers both classical and quantum communication over quantum broadcast channels and proves a superposition coding inner bound similar to our Theorem 1. Ref. [11] discusses classical communication over a bosonic broadcast channel, and Ref. [12] considers a Marton rate region for quantum communication.

In this paper, we derive two achievable rate regions for classical-quantum broadcast channels by exploiting error analysis techniques developed in the context of quantum interference channels [13], [14]. In Section III, we prove achievability of the superposition coding inner bound (Theorem 1), by using a quantum simultaneous nonunique decoder at one of the receivers. Yard *et al*. independently proved the quantum superposition coding inner bound [10], but our proof is arguably simpler and more in the spirit of its classical analogue [1]. In Section IV we prove that the quantum Marton rate region with no common message is achievable (Theorem 2). In the Marton coding scheme, the sub-channels to each receiver are essentially point-to-point, but it turns out that a technique that we call the "projector trick" seems to be necessary in our proof. We discuss open problems and give an outlook for the future in Section V.

II. Preliminaries

1) Notation: We denote classical random variables as X, U, W, whose realizations are elements of the respective finite alphabets $\mathcal{X}, \mathcal{U}, \mathcal{W}$. Let p_X, p_U, p_W denote their corresponding probability distributions. We denote quantum systems as A, B, and C and their corresponding Hilbert spaces as $\mathcal{H}^A$, $\mathcal{H}^B$, and $\mathcal{H}^C$. We represent quantum states of a system A with a density operator ρ^A, which is a positive semi-definite operator with unit trace. Let $H(A)_\rho \equiv -\text{Tr}\left\{\rho^A \log \rho^A\right\}$ denote the von Neumann entropy of the state ρ^A. A classical-quantum channel, $\mathcal{N}^{X\to B}$, is represented by the set of $|\mathcal{X}|$ possible output states $\{\rho_x^B \equiv \mathcal{N}^{X\to B}(x)\}$, meaning that a classical input of x leads to a quantum output ρ_x^B. In a communication scenario, the decoding operations performed by the receivers correspond to quantum measurements on the outputs of the channel. A quantum measurement is a positive operator-valued measure (POVM) $\{\Lambda_m\}_{m\in\{1,\ldots,|\mathcal{M}|\}}$ on the system B^n, the output of which we denote M'. To be a valid POVM, the set of $|\mathcal{M}|$ operators Λ_m must all be positive semi-definite and sum to the identity: $\Lambda_m \geq 0, \ \sum_m \Lambda_m = I$.

2) Definitions: We define a classical-quantum-quantum broadcast channel as the following map:

$$x \to \rho_x^{B_1 B_2}, \tag{1}$$

where x is a classical letter in an alphabet $\mathcal{X}$ and $\rho_x^{B_1B_2}$ is a density operator on the tensor product Hilbert space for systems B_1 and B_2. The model is such that when the sender inputs a classical letter x, Receiver 1 obtains system B_1, and Receiver 2 obtains system B_2. Since Receiver 1 does not have access to the B_2 part of the state $\rho_x^{B_1B_2}$, we model his state as $\rho_x^{B_1} = \text{Tr}_{B_2}\big[\rho_x^{B_1B_2}\big]$, where Tr_{B_2} denotes the partial trace over Receiver 2's system.

3) Information processing task: The task of communication over a broadcast channel is to use n independent instances of the channel in order to communicate with Receiver 1 at a rate R_1 and to Receiver 2 at a rate R_2. More specifically, the sender chooses a pair of messages (m_1, m_2) from message sets $\mathcal{M}_i \equiv \{1, 2, \ldots, |\mathcal{M}_i|\}$, where $|\mathcal{M}_i| = 2^{nR_i}$, and encodes these messages into an n-symbol codeword $x^n(m_1, m_2) \in \mathcal{X}^n$ suitable as input for the n channel uses.

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The output of the channel is a quantum state of the form:

$$\mathcal{N}^{\otimes n}(x^n(m_1,m_2)) \equiv \rho^{B_1^n B_2^n}_{x^n(m_1,m_2)} \quad \in \mathcal{H}^{B_1^n B_2^n}. \tag{2}$$

where $\rho^{B_1^n B_2^n}_{x^n} \equiv \rho^{B_{11}B_{21}}_{x_1} \otimes \cdots \otimes \rho^{B_{1n}B_{2n}}_{x_n}$. To decode the message m_1 intended for him, Receiver 1 performs a POVM $\{\Lambda_{m_1}\}_{m_1\in\{1,\ldots,|\mathcal{M}_1|\}}$ on the system B_1^n, the output of which we denote M_1'. Receiver 2 similarly performs a POVM $\{\Gamma_{m_2}\}_{m_2\in\{1,\ldots,|\mathcal{M}_2|\}}$ on the system B_2^n, and the random variable associated with the outcome is denoted M_2'.

An error occurs whenever either of the receivers decodes the message incorrectly. The probability of error for a particular message pair (m_1, m_2) is

$$p_e(m_1,m_2) \equiv \text{Tr}\left\{ (I - \Lambda_{m_1} \otimes \Gamma_{m_2})\, \rho^{B_1^n B_2^n}_{x^n(m_1,m_2)} \right\},$$

where the measurement operator $(I - \Lambda_{m_1} \otimes \Gamma_{m_2})$ represents the complement of the correct decoding outcome.

Definition 1. *An (n, R_1, R_2, ϵ) broadcast channel code consists of a codebook $\{x^n(m_1,m_2)\}_{m_1\in\mathcal{M}_1, m_2\in\mathcal{M}_2}$ and two decoding POVMs $\{\Lambda_{m_1}\}_{m_1\in\mathcal{M}_1}$ and $\{\Gamma_{m_2}\}_{m_2\in\mathcal{M}_2}$ such that the average probability of error $\overline{p}_e$ is bounded from above as*

$$\overline{p}_e \equiv \frac{1}{|\mathcal{M}_1||\mathcal{M}_2|} \sum_{m_1,m_2} p_e(m_1,m_2) \leq \epsilon. \tag{3}$$

A rate pair (R_1, R_2) is *achievable* if there exists an $(n, R_1-\delta, R_2-\delta, \epsilon)$ quantum broadcast channel code for all $\epsilon, \delta > 0$ and sufficiently large n.

When devising coding strategies for c-q channels, the main obstacle to overcome is the construction of a decoding POVM that correctly decodes the messages. Given a set of positive operators $\{\Pi'_m\}$ which are suitable for detecting each message, we can construct a POVM by *normalizing* them using the square-root measurement [6], [7]:

$$\Lambda_m \equiv \left(\sum_k \Pi'_k\right)^{-\frac{1}{2}} \Pi'_m \left(\sum_k \Pi'_k\right)^{-\frac{1}{2}}. \tag{4}$$

Thus, the search for a decoding POVM is reduced to the problem of finding positive operators Π'_m apt at detecting and distinguishing the output states produced by each of the possible input messages ($\text{Tr}[\Pi'_m\ \rho_m] \geq 1-\epsilon$ and $\text{Tr}[\Pi'_m\ \rho_{m'\neq m}] \leq \epsilon$).

III. Superposition coding inner bound

One possible strategy for the broadcast channel is to send a message at a rate that is low enough so that both receivers are able to decode. Furthermore, if we assume that Receiver 1 has a better reception signal, then the sender can encode a further message *superimposed* on top of the common message that Receiver 1 will be able to decode *given* the common message. The sender encodes the common message at rate R_2 using a codebook generated from a probability distribution $p_W(w)$, and the additional message for Receiver 1 at rate R_1 using a conditional codebook with distribution $p_{X|W}(x|w)$.

Theorem 1 (Superposition coding inner bound)**.** *A rate pair (R_1, R_2) is achievable for the quantum broadcast channel in (1) if it satisfies the following inequalities:*

$$R_1 \leq I(X;B_1|W)_\theta, \tag{5}$$
$$R_2 \leq I(W;B_2)_\theta, \tag{6}$$
$$R_1 + R_2 \leq I(X;B_1)_\theta, \tag{7}$$

where the above information quantities are with respect to a state $\theta^{WXB_1B_2}$ of the form

$$\sum_{w,x} p_W(w) p_{X|W}(x|w)\ |w\rangle\langle w|^W \otimes |x\rangle\langle x|^X \otimes \rho_x^{B_1B_2}. \tag{8}$$

Proof: The new idea in the proof is to exploit superposition encoding and a quantum simultaneous nonunique decoder for the decoding of the first receiver [2], [3] instead of the quantum successive decoding used in [10]. We use a standard HSW decoder for the second receiver [6], [7].

Codebook generation. We generate randomly and independently M_2 codewords $w^n(m_2)$ according to the product distribution $p_{W^n}(w^n) \equiv \prod_{i=1}^n p_W(w_i)$. For each codeword $w^n(m_2)$, we randomly and conditionally independently generates M_1 codewords $x^n(m_1,m_2)$ according to the product distribution: $p_{X^n|W^n}(x^n|w^n(m_2)) \equiv \prod_{i=1}^n p_{X|W}(x_i|w_i(m_2))$. The sender then transmits the codeword $x^n(m_1,m_2)$ if she wishes to send (m_1, m_2).

POVM Construction. We now describe the POVMs that the receivers employ in order to decode the transmitted messages. First consider the state we obtain from (8) by tracing over the B_2 system:

$$\rho^{WXB_1} = \sum_{w,x} p_W(w)\ p_{X|W}(x|w)\ |w\rangle\langle w|^W \otimes |x\rangle\langle x|^X \otimes \rho_x^{B_1}.$$

Further tracing over the X system gives

$$\rho^{WB_1} = \sum_w p_W(w)\ |w\rangle\langle w|^W \otimes \sigma_w^{B_1},$$

where $\sigma_w^{B_1} \equiv \sum_x p_{X|W}(x|w)\, \rho_x^{B_1}$. For the first receiver, we exploit a square-root decoding POVM as in (4) based on the following positive operators:

$$\Pi'_{m_1,m_2} \equiv \Pi\ \Pi_{W^n(m_2)}\ \Pi_{X^n(m_1,m_2)}\ \Pi_{W^n(m_2)}\ \Pi, \tag{9}$$

where we have made the abbreviations

$$\Pi \equiv \Pi^{B_1^n}_{\rho,\delta}, \qquad \Pi_{W^n(m_2)} \equiv \Pi^{B_1^n}_{\sigma_{W^n(m_2)},\delta},$$
$$\Pi_{X^n(m_1,m_2)} \equiv \Pi^{B_1^n}_{\rho_{X^n(m_1,m_2)},\delta}.$$

The above projectors are weakly typical projectors [5, Section 14.2.1] defined with respect to the states $\rho^{\otimes n}$, $\sigma^{B_1^n}_{W^n(m_2)}$, and $\rho^{B_1^n}_{X^n(m_1,m_2)}$.

Consider now the state in (8) as it looks from the point of view of Receiver 2. If we trace over the X and B_1 systems, we obtain the following state:

$$\rho^{WB_2} = \sum_w p_W(w)\ |w\rangle\langle w|^W \otimes \sigma_w^{B_2},$$

where $\sigma_w^{B_2} \equiv \sum_x p_{X|W}(x|w)\,\rho_x^{B_2}$. For the second receiver, we exploit a standard HSW decoding POVM that is with respect to the above state—it is a square-root measurement as in (4), based on the following positive operators:

$$\Pi'^{B_2^n}_{m_2} = \Pi^{B_2^n}_{\rho,\delta}\ \Pi^{B_2}_{\sigma_{W^n(m_2)},\delta}\ \Pi^{B_2^n}_{\rho,\delta}, \tag{10}$$

where the above projectors are weakly typical projectors defined with respect to $\rho^{\otimes n}$ and $\sigma^{B_2^n}_{W^n(m_2)}$.

Error analysis. We now analyze the expectation of the average error probability for the first receiver with the POVM defined by (4) and (9):

$$\mathop{\mathbb{E}}_{X^n,W^n}\left\{\frac{1}{M_1M_2}\sum_{m_1,m_2}\text{Tr}\left\{\left(I-\Gamma^{B_1^n}_{m_1,m_2}\right)\rho^{B_1}_{X^n(m_1,m_2)}\right\}\right\}$$
$$=\frac{1}{M_1M_2}\sum_{m_1,m_2}\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\left(I-\Gamma^{B_1^n}_{m_1,m_2}\right)\rho^{B_1}_{X^n(m_1,m_2)}\right\}\right\}.$$

Due to the above exchange between the expectation and the average and the symmetry of the code construction (each codeword is selected randomly and independently), it suffices to analyze the expectation of the average error probability for the first message pair $(m_1=1, m_2=1)$, i.e., the last line above is equal to $\mathbb{E}_{X^n,W^n}\left\{\text{Tr}\left\{\left(I-\Gamma^{B_1^n}_{1,1}\right)\rho^{B_1}_{X^n(1,1)}\right\}\right\}$. Using the Hayashi-Nagaoka operator inequality (Lemma 3 in the appendix of Ref. [15]), we obtain the following upper bound on this term:

$$2\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\left(I-\Pi'_{1,1}\right)\rho^{B_1}_{X^n(1,1)}\right\}\right\}$$
$$+4\sum_{(m_1,m_2)\neq(1,1)}\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\Pi'_{m_1,m_2}\rho^{B_1}_{X^n(1,1)}\right\}\right\}. \tag{11}$$

We begin by bounding the term in the first line above. Consider the following chain of inequalities:

$$\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\Pi'_{1,1}\rho^{B_1}_{X^n(1,1)}\right\}\right\}$$
$$=\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\Pi\ \Pi_{W^n(1)}\Pi_{X^n(1,1)}\ \Pi_{W^n(1)}\ \Pi\ \rho^{B_1}_{X^n(1,1)}\right\}\right\}$$
$$\geq\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\Pi_{X^n(1,1)}\ \ \rho^{B_1}_{X^n(1,1)}\right\}\right\}$$
$$-\mathop{\mathbb{E}}_{X^n,W^n}\left\{\left\|\rho^{B_1}_{X^n(1,1)}-\Pi\ \rho^{B_1}_{X^n(1,1)}\ \Pi\right\|_1\right\}$$
$$-\mathop{\mathbb{E}}_{X^n,W^n}\left\{\left\|\rho^{B_1}_{X^n(1,1)}-\Pi_{W^n(1)}\ \rho^{B_1}_{X^n(1,1)}\ \Pi_{W^n(1)}\right\|_1\right\}$$
$$\geq 1-\epsilon-4\sqrt{\epsilon},$$

where the first inequality follows from the inequality

$$\text{Tr}\{\Lambda\rho\}\leq\text{Tr}\{\Lambda\sigma\}+\|\rho-\sigma\|_1, \tag{12}$$

which holds for all ρ, σ, and Λ such that $0\leq\rho,\sigma,\Lambda\leq I$. The second inequality follows from the Gentle Operator Lemma for ensembles (see Lemma 2 in the appendix of Ref. [15]) and the properties of typical projectors for sufficiently large n.

We now focus on bounding the term in the second line of (11). We can expand this term as follows:

$$\sum_{m_1\neq1}\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\Pi'_{m_1,1}\ \rho^{B_1}_{X^n(1,1)}\right\}\right\}$$
$$+\sum_{\substack{m_1,\\ m_2\neq1}}\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\Pi'_{m_1,m_2}\ \rho^{B_1}_{X^n(1,1)}\right\}\right\}. \tag{13}$$

Consider the term in the first line of (13):

$$\sum_{m_1\neq1}\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\Pi'_{m_1,1}\ \rho^{B_1}_{X^n(1,1)}\right\}\right\}$$
$$=\sum_{m_1\neq1}\mathop{\mathbb{E}}_{X^n,W^n}\text{Tr}\left\{\Pi\ \Pi_{W^n(1)}\ \Pi_{X^n(m_1,1)}\ \Pi_{W^n(1)}\ \Pi\ \rho^{B_1}_{X^n(1,1)}\right\}$$
$$\leq K\sum_{m_1\neq1}\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left[\Pi\ \Pi_{W^n(1)}\rho_{X^n(m_1,1)}\Pi_{W^n(1)}\Pi\ \rho^{B_1}_{X^n(1,1)}\right]\right\}$$
$$=K\sum_{m_1\neq1}\mathop{\mathbb{E}}_{W^n}\left\{\text{Tr}[\Pi_{W^n(1)}\mathop{\mathbb{E}}_{X^n|W^n}\left\{\rho_{X^n(m_1,1)}\right\}\ \Pi_{W^n(1)}\right.$$
$$\left.\Pi\mathop{\mathbb{E}}_{X^n|W^n}\left\{\rho^{B_1}_{X^n(1,1)}\right\}\Pi\ ]\right\}$$
$$=K\sum_{m_1\neq1}\mathop{\mathbb{E}}_{W^n}\left\{\text{Tr}\left\{\Pi\ \Pi_{W^n(1)}\sigma_{W^n(1)}\ \Pi_{W^n(1)}\ \Pi\ \sigma_{W^n(1)}\right\}\right\}$$
$$\leq K\,2^{-n[H(B_1|W)-\delta]}\sum_{m_1\neq1}\mathop{\mathbb{E}}_{W^n}\left\{\text{Tr}\left\{\Pi\ \Pi_{W^n(1)}\ \Pi\ \sigma_{W^n(1)}\right\}\right\}$$
$$\leq K\,2^{-n[H(B_1|W)-\delta]}\sum_{m_1\neq1}\mathop{\mathbb{E}}_{W^n}\left\{\text{Tr}\left\{\sigma_{W^n(1)}\right\}\right\}$$
$$\leq 2^{-n[I(X;B_1|W)-2\delta]}\ M_1,$$

where we define $K\equiv2^{n[H(B_1|WX)+\delta]}$. The first inequality is due to the *projector trick* inequality [16], [14], [13] which states that

$$\Pi_{X^n(m_1,1)}\leq2^{n[H(B_1|WX)+\delta]}\,\rho^{B_1}_{X^n(m_1,1)}. \tag{14}$$

The second inequality follows from the properties of typical projectors: $\Pi_{W^n(1)}\sigma_{W^n(1)}\ \Pi_{W^n(1)}\leq2^{-n[H(B_1|W)-\delta]}\Pi_{W^n(1)}$.

Now consider the term in the second line of (13):

$$\sum_{\substack{m_1,\\ m_2\neq1}}\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left\{\Pi'_{m_1,m_2}\ \rho^{B_1}_{X^n(1,1)}\right\}\right\}$$
$$=\sum_{\substack{m_1,\\ m_2\neq1}}\mathop{\mathbb{E}}_{X^n,W^n}\left\{\text{Tr}\left[\Pi\Pi_{W^n(m_2)}\Pi_{X^n(m_1,m_2)}\Pi_{W^n(m_2)}\Pi\ \rho^{B_1}_{X^n(1,1)}\right]\right\}$$
$$=\sum_{\substack{m_1,\\ m_2\neq1}}\text{Tr}\left[\mathop{\mathbb{E}}_{X^n,W^n}\left\{\Pi_{W^n(m_2)}\ \Pi_{X^n(m_1,m_2)}\ \Pi_{W^n(m_2)}\right\}\right.$$
$$\left.\Pi\mathop{\mathbb{E}}_{X^n,W^n}\left\{\rho^{B_1}_{X^n(1,1)}\right\}\Pi\ \right]$$
$$=\sum_{\substack{m_1,\\ m_2\neq1}}\text{Tr}\left\{\mathop{\mathbb{E}}_{X^n,W^n}\left\{\Pi_{W^n(m_2)}\Pi_{X^n(m_1,m_2)}\Pi_{W^n(m_2)}\right\}\ \Pi\rho^{\otimes n}\Pi\right\}$$
$$\leq2^{-n[H(B_1)-\delta]}\sum_{\substack{m_1,\\ m_2\neq1}}\text{Tr}\left[\mathop{\mathbb{E}}_{X^n,W^n}\left\{\Pi_{W^n(m_2)}\Pi_{X^n(m_1,m_2)}\Pi_{W^n(m_2)}\right\}\Pi\right]$$
$$=2^{-n[H(B_1)-\delta]}\sum_{\substack{m_1,\\ m_2\neq1}}\mathop{\mathbb{E}}_{X^n,W^n}\text{Tr}\left[\Pi_{X^n(m_1,m_2)}\Pi_{W^n(m_2)}\Pi\Pi_{W^n(m_2)}\right]$$

$$\leq 2^{-n[H(B_1)-\delta]} \sum_{m_2\neq 1,\ m_1} \mathbb{E}_{X^n,W^n} \left\{\text{Tr}\left\{\Pi_{X^n(m_1,m_2)}\right\}\right\}$$
$$\leq 2^{-n[H(B_1)-\delta]}\ 2^{n[H(B_1|WX)+\delta]}\ M_1\ M_2$$
$$= 2^{-n[I(WX;B_1)-2\delta]}\ M_1\ M_2$$
$$= 2^{-n[I(X;B_1)-2\delta]}\ M_1\ M_2.$$

The equality $I(WX;B_1) = I(X;B_1)$ follows from the way the codebook is constructed (i.e., the Markov chain $W - X - B$), as discussed also in [14]. This completes the error analysis for the first receiver.

The proof for the second receiver is analogous to the point-to-point HSW theorem. The appendix of Ref. [15] gives the details for this and ties the coding theorem together so that the sender and two receivers can agree on a strategy that has asymptotically vanishing error probability in the large n limit. ∎

IV. Marton coding scheme

We now prove that the Marton inner bound is achievable for quantum broadcast channels. The Marton scheme depends on auxiliary random variables U_1 and U_2, *binning*, and the properties of strongly typical sequences and projectors.

Theorem 2 (Marton inner bound)**.** *Let $\{\rho_x^{B_1B_2}\}$ be a classical-quantum broadcast channel and $x = f(u_1,u_2)$ be a deterministic function. The following rate region is achievable:*

$$\begin{aligned} R_1 &\leq I(U_1;B_1)_\theta, \\ R_2 &\leq I(U_2;B_2)_\theta, \\ R_1+R_2 &\leq I(U_1;B_1)_\theta + I(U_2;B_2)_\theta - I(U_1;U_2)_\theta, \end{aligned} \tag{15}$$

where the information quantities are with respect to the state:

$$\theta^{U_1U_2B_1B_2} = \sum_{u_1,u_2} p(u_1,u_2)|u_1\rangle\langle u_1|^{U_1}\otimes|u_2\rangle\langle u_2|^{U_2}\otimes\rho^{B_1B_2}_{f(u_1,u_2)}.$$

Proof: Consider the classical-quantum broadcast channel $\{\mathcal{N}(x) \equiv \rho_x^{B_1B_2}\}$, and a deterministic mixing function: $f : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{X}$. Using the mixing function as a pre-coder to the broadcast channel $\mathcal{N}$, we obtain a channel $\mathcal{N}'$ defined as:

$$\mathcal{N}'(u_1,u_2) \equiv \rho^{B_1B_2}_{f(u_1,u_2)} \equiv \rho^{B_1B_2}_{u_1,u_2}. \tag{16}$$

Codebook construction. Define two auxiliary indices $\ell_1 \in [1,\ldots,L_1]$, $L_1 = 2^{n[I(U_1;B_1)-\delta]}$ and $\ell_2 \in [1,\ldots,L_2]$, $L_2 = 2^{n[I(U_2;B_2)-\delta]}$. For each ℓ_1 generate an i.i.d. random sequence $u_1^n(\ell_1)$ according to $p_{U_1^n}(u_1^n)$. Similarly we choose L_2 random i.i.d. sequences $u_2^n(\ell_2)$ according to $p_{U_2^n}(u_2^n)$. Partition the sequences $u_1^n(\ell_1)$ into 2^{nR_1} different bins B_{m_1}. Similarly, partition the sequences $u_2^n(\ell_2)$ into 2^{nR_2} bins C_{m_2}. For each message pair (m_1,m_2), the sender selects a sequence $\left(u_1^n(\ell_1),u_2^n(\ell_2)\right) \in (B_{m_1}\times C_{m_2}) \cap \mathcal{A}^n_{p_{U_1U_2},\delta}$, such that each sequence is taken from the appropriate bin and the sender demands that they are strongly jointly-typical (otherwise declaring failure). The codebook $x^n(m_1,m_2)$ is deterministically constructed from $\left(u_1^n(\ell_1),u_2^n(\ell_2)\right)$ by applying the function $x_i = f(u_{1i},u_{2i})$.

Transmission. Let (ℓ_1,ℓ_2) denote the pair of indices of the joint sequence $(u_1^n(\ell_1),u_2^n(\ell_2))$ which was chosen as the codeword for message (m_1,m_2). Expressed in terms of these indices the output of the channel is

$$\rho^{B_1^nB_2^n}_{u_1^n(\ell_1),u_2^n(\ell_2)} = \bigotimes_{i\in[n]} \rho^{B_1B_2}_{f(u_{1i}(\ell_1),u_{2i}(\ell_2))} \equiv \rho_{\ell_1,\ell_2}. \tag{17}$$

Define the following states:

$$\omega_{u_1}^{B_1} \equiv \sum_{u_2} p_{U_2|U_1}(u_2|u_1)\rho^{B_1}_{u_1,u_2}, \quad \bar{\rho} \equiv \sum_{u_1} p(u_1)\omega^{B_1}_{u_1}. \tag{18}$$

Decoding. The detection POVM for Receiver 1, $\{\Lambda_{\ell_1}\}_{\ell_1\in[1,\ldots,L_1]}$, is constructed by using the square-root measurement as in (4) based on the following combination of strongly typical projectors:

$$\Pi'_{\ell_1} \equiv \Pi^n_{\bar{\rho},\delta}\ \Pi_{u_1^n(\ell_1)}\ \Pi^n_{\bar{\rho},\delta}. \tag{19}$$

The projectors $\Pi_{u_1^n(\ell_1)}$ and $\Pi^n_{\bar{\rho},\delta}$ are defined with respect to the states $\omega_{u_1^n(\ell_1)}$ and $\bar{\rho}^{\otimes n}$ given in (18). Note that we use *strongly* typical projectors in this case [5, Section 14.2.3]. Knowing ℓ_1 and the binning scheme, Receiver 1 can deduce the message m_1 from the bin index. Receiver 2 uses a similar decoding strategy to obtain ℓ_2 and infer m_2.

Error analysis. An error occurs if one (or more) of the following error events occurs.

- E_0: An encoding error occurs whenever there is no jointly typical sequence in $B_{m_1}\times C_{m_2}$ for some message pair (m_1,m_2).
- E_1: A decoding error occurs at Receiver 1 if $L_1' \neq \ell_1$.
- E_2: A decoding error occurs at Receiver 2 if $L_2' \neq \ell_2$.

The probability of an encoding error E_0 is bounded like in the classical Marton scheme [4], [1], [17]. To see this, we use Cover's counting argument [17]. The probability that two random sequences u_1^n, u_2^n chosen according to the marginals are jointly typical is $2^{-nI(U_1;U_2)}$ and since there are on average $2^{n[I(U_1;B_1)-R_1]}$ and $2^{n[I(U_2;B_2)-R_2]}$ sequences in each bin, the expected number of jointly-typical sequences that can be constructed from each combination of bins is

$$2^{n[I(U_1;B_1)-R_1]}2^{n[I(U_2;B_2)-R_2]}2^{-nI(U_1;U_2)}. \tag{20}$$

Thus, if we choose $R_1+R_2+\delta \leq I(U_1;B_1)+I(U_2;B_2)-I(U_1;U_2)$, then the expected number of strongly jointly-typical sequences in $B_{m_1}\times C_{m_2}$ is much larger than one.

To bound the probability of error event E_1, we use the Hayashi-Nagaoka operator inequality (Lemma 3 in the appendix of Ref. [15]):

$$\Pr(E_1) = \frac{1}{L_1}\sum_{\ell_1} \text{Tr}\ \left[(I-\Lambda_{\ell_1})\rho_{\ell_1,\ell_2}\right]$$
$$\leq \frac{1}{L_1L_2}\sum_{\ell_1}\Bigg(2\underbrace{\text{Tr}\left[(I-\Pi^n_{\bar{\rho},\delta}\Pi_{u_1^n(\ell_1)}\Pi^n_{\bar{\rho},\delta})\rho_{\ell_1,\ell_2}\right]}_{(T1)} + 4\underbrace{\sum_{\ell_1'\neq\ell_1}\text{Tr}\left[\Pi^n_{\bar{\rho},\delta}\Pi_{u_1^n(\ell_1')}\Pi^n_{\bar{\rho},\delta}\rho_{\ell_1,\ell_2}\right]}_{(T2)}\Bigg).$$

Consider the following lemma [5, Property 14.2.7].

Lemma 1. *The state ρ_{ℓ_1,ℓ_2} is well supported by both the averaged state projector:* $\mathrm{Tr}\big[\Pi^n_{\bar{\rho},\delta}\ \rho_{\ell_1,\ell_2}\big] \geq 1-\epsilon$, $\forall \ell_1, \ell_2$, *and the $\omega^{B_1}_{u_1}$ conditionally typical projector:* $\mathrm{Tr}\big[\Pi_{u_1^n(\ell_1)}\ \rho_{\ell_1,\ell_2}\big] \geq 1-\epsilon$, $\forall \ell_2$, *when $u_1^n(\ell_1)$ and $u_2^n(\ell_2)$ are strongly jointly typical.*

To bound the first term (T1), we use the following argument:

$$\begin{aligned}1-(T1) &= \mathrm{Tr}\big[\Pi^n_{\bar{\rho},\delta}\Pi_{u_1^n(\ell_1)}\Pi^n_{\bar{\rho},\delta}\ \rho_{\ell_1,\ell_2}\big] \\ &= \mathrm{Tr}\big[\Pi_{u_1^n(\ell_1)}\ \Pi^n_{\bar{\rho},\delta}\rho_{\ell_1,\ell_2}\Pi^n_{\bar{\rho},\delta}\big] \\ &\geq \mathrm{Tr}\big[\Pi_{u_1^n(\ell_1)}\ \rho_{\ell_1,\ell_2}\big] - \|\Pi^n_{\bar{\rho},\delta}\rho_{\ell_1,\ell_2}\Pi^n_{\bar{\rho},\delta} - \rho_{\ell_1,\ell_2}\|_1 \\ &\geq (1-\epsilon) - 2\sqrt{\epsilon}, \end{aligned} \tag{21}$$

where the inequalities follow from (12) and Lemma 1. This use of Lemma 1 demonstrates why the Marton coding scheme selects the sequences $u_1^n(\ell_1)$ and $u_2^n(\ell_2)$ such that they are strongly jointly typical.

To bound the second term, we begin by applying a variant of the projector trick from (14). In the below, note that the expectation $\mathbb{E}_{U_1,U_2}$ over the random code is with respect to the product distribution $p_{U_1^n}(u_1^n)p_{U_2^n}(u_2^n)$:

$$\mathop{\mathbb{E}}_{U_1,U_2}\{(T2)\} = \mathop{\mathbb{E}}_{U_1,U_2}\left\{\sum_{\ell_1'\neq\ell_1}\mathrm{Tr}\big[\Pi^n_{\bar{\rho},\delta}\Pi_{U_1^n(\ell_1')}\Pi^n_{\bar{\rho},\delta}\ \rho_{\ell_1,\ell_2}\big]\right\}$$

$$\leq 2^{n[H(B_1|U_1)+\delta]}\mathop{\mathbb{E}}_{U_1,U_2}\left\{\sum_{\ell_1'\neq\ell_1}\mathrm{Tr}\big[\Pi^n_{\bar{\rho},\delta}\ \omega_{\ell_1'}\ \Pi^n_{\bar{\rho},\delta}\ \rho_{\ell_1,\ell_2}\big]\right\}.$$

We continue the proof using averaging over the choice of codebook and the properties of typical projectors:

$$\begin{aligned}&= 2^{n[H(B_1|U_1)+\delta]}\mathop{\mathbb{E}}_{U_2}\sum_{\ell_1'\neq\ell_1}\mathrm{Tr}\left[\Pi^n_{\bar{\rho},\delta}\ \mathop{\mathbb{E}}_{U_1}\{\omega_{\ell_1'}\}\ \Pi^n_{\bar{\rho},\delta}\mathop{\mathbb{E}}_{U_1}\{\rho_{\ell_1,\ell_2}\}\right] \\ &= 2^{n[H(B_1|U_1)+\delta]}\mathop{\mathbb{E}}_{U_2}\sum_{\ell_1'\neq\ell_1}\mathrm{Tr}\left[\Pi^n_{\bar{\rho},\delta}\ \bar{\rho}\ \Pi^n_{\bar{\rho},\delta}\ \mathop{\mathbb{E}}_{U_1}\{\rho_{\ell_1,\ell_2}\}\right] \\ &\leq 2^{n[H(B_1|U_1)+\delta]}2^{-n[H(B_1)-\delta]}\mathop{\mathbb{E}}_{U_1,U_2}\sum_{\ell_1'\neq\ell_1}\mathrm{Tr}\big[\Pi^n_{\bar{\rho},\delta}\ \rho_{\ell_1,\ell_2}\big] \\ &\leq 2^{n[H(B_1|U_1)+\delta]}2^{-n[H(B_1)-\delta]}\mathop{\mathbb{E}}_{U_1,U_2}\sum_{\ell_1'\neq\ell_1}1 \\ &\leq L_1 2^{-n[I(U_1;B_1)-2\delta]}.\end{aligned}$$

Therefore, if we choose $2^{nR_1} = L_1 \leq 2^{n[I(U_1;B_1)-3\delta]}$, the probability of error will go to zero in the asymptotic limit of many channel uses. The analysis of the event E_2 is similar. ■

V. Conclusion

We have proved quantum generalizations of the superposition coding inner bound [2], [3] and the Marton rate region with no common message [4]. The key ingredient in both proofs was the use of the projector trick. A natural followup question would be to combine the two strategies to obtain the Marton coding scheme with a common message.

A much broader goal would be to extend all of network information theory to the study of quantum channels. To accomplish this goal, it would be helpful to have a powerful tool that generalizes El Gamal and Kim's classical packing lemma [1] to the quantum domain. The packing lemma is sufficient to prove all of the known coding theorems in network information theory. At the moment, it is not clear to us whether such a tool exists for the quantum case, but evidence in favor of its existence is that 1) one can prove the HSW coding theorem by using conditionally typical projectors only [5, Exercise 19.3.5], 2) we have solved the quantum simultaneous decoding conjecture for the case of two senders [13], [14], and 3) we have generalized two important coding theorems in the current paper (with proofs somewhat similar to the classical proofs). Ideally, such a tool would allow quantum information theorists to prove quantum network coding theorems by appealing to it, rather than having to analyze each coding scheme in detail on a case by case basis.

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