Trading Classical Communication, Quantum Communication, and Entanglement in Quantum Shannon Theory

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***Abstract*—In this paper, we give tradeoffs between classical communication, quantum communication, and entanglement for processing information in the Shannon-theoretic setting. We first prove a "unit-resource" capacity theorem that applies to the scenario where only the above three noiseless resources are available for consumption or generation. The optimal strategy mixes the three fundamental protocols of teleportation, superdense coding, and entanglement distribution. We then provide an achievable rate region and a matching multiletter converse for the "direct-static" capacity theorem. This theorem applies to the scenario where a large number of copies of a noisy bipartite state are available (in addition to consumption or generation of the above three noiseless resources). Our coding strategy involves a protocol that we name the *classically assisted state redistribution protocol* and the three fundamental protocols. We finally provide an achievable rate region and a matching multiletter converse for the "direct-dynamic" capacity theorem. This theorem applies to the scenario where a large number of uses of a noisy quantum channel are available in addition to the consumption or generation of the three noiseless resources. Our coding strategy combines the *classically enhanced father protocol* with the three fundamental unit protocols.**

***Index Terms*—Classical communication, direct-dynamic capacity theorem, direct-static capacity theorem, entanglement, entanglement-assisted quantum coding, quantum communication, quantum Shannon theory.**

I. Introduction

The publication of Shannon's classic article in 1948 formally marks the beginning of information theory [1]. Shannon's article states two fundamental theorems: the source coding theorem and the channel coding theorem. The source coding theorem concerns processing of a *static* resource—an information source that emits a symbol from an alphabet where each symbol occurs with some probability. The proof of the theorem appeals to the asymptotic setting where many copies of the static resource are available, i.e., the information source emits a large number of symbols. The result of the source coding theorem is a tractable lower bound on the compressibility of the static resource. On the other hand, the channel coding theorem applies to a *dynamic* resource. An example of a dynamic resource is a noisy bit-flip channel that flips each input bit with a certain probability. The proof of the channel coding theorem again appeals to the asymptotic setting where a sender consumes a large number of independent and identically distributed (i.i.d.) uses of the channel to transmit information to a receiver. The result of the channel coding theorem is a tractable upper bound on the reliable transmission rate of the dynamic resource.

Manuscript received January 27, 2009; revised April 15, 2010. Date of current version August 18, 2010. The work of M. M. Wilde was supported by the National Research Foundation and Ministry of Education, Singapore, the Science Applications International Corporation under Internal Research and Development Grant SAIC-1669, and the MDEIE (Québec) PSR-SIIRI International Collaboration Grant. The material in this paper was presented in part at the 4th Workshop on the Theory of Quantum Computation, Communication, and Cryptography, Waterloo, ON, Canada, May 2009.

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Communicated by P. Hayden, Associate Editor for Quantum Information Theory.

Color versions of Figures 1–10 in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TIT.2010.2054532

Quantum Shannon theory has emerged in recent years as the quantum generalization of Shannon's information theory. Schumacher established a quantum source coding theorem that is a "quantized" version of Shannon's source coding theorem [2], [3]. Schumacher's static resource is a quantum information source that emits a given quantum state with a certain probability. Holevo, Schumacher, and Westmoreland (HSW) followed by proving that the Holevo information of a quantum channel is an achievable rate for transmitting classical information over a noisy quantum channel [4], [5]. Lloyd, Shor, and Devetak then proved that the coherent information of a quantum channel is an achievable rate for transmitting quantum data over a noisy quantum channel [6]–[8]. Both of these quantum channel coding theorems exploit a dynamic resource—a noisy quantum channel that connects a sender to a receiver. Entanglement is a *static resource* shared between a sender and a receiver. It is "static" because a sender and a receiver cannot exploit entanglement alone to generate either classical communication or quantum communication or both. However, they can exploit entanglement and classical communication to communicate quantum information—this protocol is the well-known teleportation (TP) protocol [9]. The superdense coding (SD) protocol [10] doubles the classical capacity of a noiseless quantum channel by exploiting entanglement in addition to the use of the noiseless quantum channel. These two protocols and others demonstrate that entanglement is a valuable resource in quantum information processing.

Several researchers have shown how to process entanglement in the asymptotic setting where a large number of identical copies of an entangled state or a large number of independent uses of a noisy channel are available to generate entanglement.

Bennett *et al.* proved a coding theorem for entanglement concentration that determines how much entanglement (in terms of maximally entangled states) a sender and a receiver can generate from pure bipartite states [11]. The reverse problem of entanglement dilution [12]–[14] shows that entanglement is not an interconvertible static resource (i.e., simulating pure bipartite states from maximally entangled states requires a sublinear amount of classical communication). A dynamic resource can also generate entanglement. Devetak proved a coding theorem that determines how much entanglement a sender and a receiver can generate by sending quantum states through a noisy quantum channel [8].

Quantum Shannon theory began with the aforementioned single-resource coding theorems [2]–[8], [11]–[15] and has advanced to include double-resource coding theorems—their corresponding protocols either generate two different resources or they generate one resource with the help of another [16]–[25]. The result of each of these scenarios was an achievable 2-D tradeoff region for the resources involved in the protocols. Quantum information theorists have organized many of the existing protocols into a family tree [23], [25], [26]. Furthermore, Devetak *et al.* proposed the resource inequality framework that establishes many classical and quantum coding theorems as interconversions between *nonlocal information resources* [23], [25]. The language of resource inequalities provides structural insights into the relationships between coding theorems in quantum Shannon theory and greatly simplifies the development of new coding schemes. An example of one of the resource tradeoffs is the so-called "father" achievable rate region. The father protocol exploits a noisy quantum channel and shared noiseless entanglement to generate noiseless quantum communication. The father achievable rate region illustrates tradeoffs between entanglement consumption and quantum communication.

In this paper, we advance quantum Shannon theory to the triple resource setting by giving the full triple tradeoffs for both the static and dynamic scenarios. This triple tradeoff solution represents one of the most general scenarios considered in quantum Shannon theory. Here, we study the interplay of the most important noiseless resources in the theory of quantum communication: namely, classical communication, quantum communication, and entanglement, with general noisy resources.

The noisy static resource that we consider here is a shared noisy bipartite state, and the dynamic resource that we consider is a noisy quantum channel. We again appeal to the asymptotic setting where a large number of independent copies or uses of the respective static or dynamic noisy resource are available. For both the static and dynamic scenarios, we assume that the sender and the receiver either consume or generate noiseless classical communication, noiseless quantum communication, and noiseless entanglement in addition to the consumption of the noisy resource. The result is a 3-D achievable rate region that gives tradeoffs for the three noiseless resources in both the static and dynamic scenarios. The rate of a noiseless resource is negative if a protocol consumes the corresponding resource, and its rate is positive if a protocol generates the resource. The above interpretation of a negative rate first appeared in [27] and [28] with the state merging protocol and with subsequent appearance, for example, in [25], [29], and [30]. This paper's solution for the static and dynamic scenarios contains both negative and positive rates.

Our current formulas characterizing the triple tradeoff capacity regions are alas of a "multiletter" nature, a problem that plagues many results in quantum Shannon theory. A multiletter formula is one that involves an intractable optimization over an arbitrary number of uses of a channel or a state, as opposed to a more desirable "single-letter" formula that involves a tractable optimization over a single use of a channel or state. In principle, a multiletter characterization is an optimal solution, but the multiletter nature of our characterization of the capacity region implies that there may be a slight room for improvement in the formulas when considering an optimization over a finite number of uses—sometimes suboptimal protocols can lead to an optimal characterization of a capacity region when taking the limit over an arbitrary number of uses of a channel or a state.

Despite the multiletter nature of our characterization in the general case, we can find examples of shared states and quantum channels for which the regions single-letterize. We show that the static region single-letterizes for the special case of an "erased state," a state that two parties obtain by sending one half of a maximally entangled Bell state through a quantum erasure channel. Our argument for single-letterization is similar to an argument we presented in [31] for the erasure channel. We also show that the dynamic region single-letterizes for the case of a qubit dephasing channel. This proof builds on earlier work in [31] and [32] to show that single-letterization holds. In a later work [33], we build on the efforts in [31] and [32] to give a concise, direct argument for single-letterization of the dynamic capacity region for the full class of Hadamard channels.

An interesting aspect of this paper is that we employ basic topological arguments and *reductio ad absurdum* arguments in the multiletter converse proofs of the triple tradeoff capacity theorems. To our knowledge, the mathematical techniques that we use are different from prior techniques in the classical information theory literature or the quantum Shannon theoretic literature (though, there are some connections to the techniques in [25]). Many times, we apply well-known results in quantum Shannon theory to reduce a converse proof to that of a previously known protocol (for example, we apply the well-known result that forward classical communication does not increase the quantum capacity of a quantum channel [16], [34]). For the constructive part of the coding theorems, there is no need to employ asymptotic arguments such as typical subspace techniques [35] or the operator Chernoff bound [36] because the resource inequality framework is sufficient to prove the coding theorems.

One benefit of the direct-dynamic capacity theorem is that we are able to answer a question concerning the use of entanglement-assisted coding [37]–[45] versus the use of TP. We show exactly the context in which entanglement-assisted coding is superior to mere TP. We consider this result an important corollary of the results in this paper because it is rare that quantum Shannon theory gives insight into practical error correction schemes.

We structure this paper as follows. In the next section, we establish some definitions and notation that prove useful for later sections. Section III briefly summarizes our three main theorems: the unit resource capacity theorem, the direct-static capacity theorem, and the direct-dynamic capacity theorem. In Section IV, we prove the optimality of a unit resource capacity region. The unit resource capacity region does not include a static or dynamic resource, but includes the resources of noiseless classical communication, noiseless quantum communication, and noiseless entanglement only. The unit resource capacity theorem shows that a mixed strategy combining TP [9], SD [10], and entanglement distribution (ED) [23] is optimal whenever a static or dynamic noisy resource is not available. Section V includes a discussion of the classically assisted state redistribution protocol. Section VI states and proves the direct-static capacity theorem. This theorem determines the tradeoffs between the three noiseless resources when a noisy static resource is available. Section VII states and proves the direct-dynamic capacity theorem. This theorem determines the triple tradeoffs when a noisy dynamic resource is available. Section VIII gives examples of states and channels for which our formulas for the respective direct-static and direct-dynamic capacity theorems admit single-letter characterizations. We end with a discussion of the results in this paper and future open problems.

II. Definitions and Notation

We first establish some notation before proceeding to the main theorems. We review the notation for the three noiseless unit resources and that for resource inequalities. We establish some notation for handling geometric objects such as lines, quadrants, and octants in the 3-D space of classical communication, quantum communication, and entanglement.

The three fundamental resources are noiseless classical communication, noiseless quantum communication, and noiseless entanglement. Let $[c \rightarrow c]$ denote one *cbit* of noiseless forward classical communication, let $[q \rightarrow q]$ denote one *qubit* of noiseless forward quantum communication, and let $[qq]$ denote one *ebit* of shared noiseless entanglement [23], [25]. The ebit is a maximally entangled state

$$\left|\Phi^{+}\right\rangle^{AB} \equiv \frac{\left(|00\rangle^{AB}+|11\rangle^{AB}\right)}{\sqrt{2}}$$

shared between two parties A and B who bear the respective names Alice and Bob. The ebit $[qq]$ is a unit static resource and both the cbit $[c \rightarrow c]$ and the qubit $[q \rightarrow q]$ are unit dynamic resources.

We consider two noisy resources: a noisy static resource and a noisy dynamic resource. Let ρ^{AB} denote the noisy static resource: a noisy bipartite state shared between Alice and Bob. Let $\mathcal{N}^{A' \rightarrow B}$ denote a noisy dynamic resource: a noisy quantum channel that connects Alice to Bob. Throughout this paper, the dynamic resource $\mathcal{N}^{A' \rightarrow B}$ is a completely positive and trace-preserving (CPTP) map that takes density operators in the Hilbert space of Alice's system A' to Bob's system B.

Resource inequalities are a compact, yet rigorous, way to state coding theorems in quantum Shannon theory [23], [25]. In this paper, we formulate resource inequalities that consume the above noisy resources and either consume or generate the noiseless resources. An example from [23] and [25] is the following "mother" resource inequality:

$$\langle\rho^{AB}\rangle + |Q|\,[q \rightarrow q] \geq E\,[qq].$$

It states that a large number n of copies of the state ρ^{AB} and $n\,|Q|$ uses of a noiseless qubit channel are sufficient to generate nE ebits of entanglement while tolerating an arbitrarily small error in the fidelity of the produced ebits. The rates Q and E of respective qubit channel consumption and entanglement generation are entropic quantities

$$|Q| = \frac{1}{2} I\left(A;E\right)$$
$$E = \frac{1}{2} I\left(A;B\right).$$

See [46] for definitions of entropy and mutual information. The entropic quantities are with respect to a state $|\psi\rangle^{EAB}$ where $|\psi\rangle^{EAB}$ is a purification of the noisy static resource state ρ^{AB} and E is the purifying reference system (it should be clear when E refers to the purifying system and when it refers to the rate of entanglement generation). We take the convention that the rate Q is negative and E is positive because the protocol consumes quantum communication and generates entanglement (this convention is the same as in [25] and [27]–[30]).

We make several geometric arguments throughout this paper because the static and dynamic capacity regions lie in a 3-D space with points that are rate triples (C,Q,E). C represents the rate of classical communication, Q the rate of quantum communication, and E the rate of entanglement consumption or generation. Let L denote a line, Q a quadrant, and O an octant in this space (it should be clear from context whether Q refers to quantum communication or "quadrant"). For example, L^{-00} denotes a line going in the direction of negative classical communication

$$L^{-00} \equiv \{\alpha\left(-1,0,0\right) : \alpha \geq 0\}.$$

Q^{0+-} denotes the quadrant where there is zero classical communication, generation of quantum communication, and consumption of entanglement

$$Q^{0+-} \equiv \{\alpha\left(0,1,0\right) + \beta\left(0,0,-1\right) : \alpha,\beta \geq 0\}.$$

O^{+-+} denotes the octant where there is generation of classical communication, consumption of quantum communication, and generation of entanglement

$$O^{+-+} \equiv \left\{ \begin{array}{c} \alpha\left(1,0,0\right) + \beta\left(0,-1,0\right) + \gamma\left(0,0,1\right) \\ : \alpha,\beta,\gamma \geq 0 \end{array} \right\}.$$

It proves useful to have a "set addition" operation between two regions A and B

$$A + B \equiv \{a + b : a \in A, b \in B\}.$$

The following relations hold:

$$Q^{0+-} = L^{0+0} + L^{00-}$$
$$O^{+-+} = L^{+00} + L^{0-0} + L^{00+}$$

by using the above definition. Set addition of the same line gives the line itself, e.g., $L^{+00} + L^{+00} = L^{+00}$. This set equality holds because of the definition of set addition and the definition of the line. A similar result also holds for addition of the same quadrant or octant. We define the set subtraction of two regions A and B as follows:

$$A - B \equiv \{a - b : a \in A, b \in B\}.$$

According to this definition, it follows that $L^{+00} \subseteq L^{+00} - L^{+00} = L^{\pm 00}$ where $L^{\pm 00}$ represents the full line of classical communication.

III. Summary of Results

We first provide an accessible overview of the main results in this paper. The interested reader can then delve into later sections of the paper for mathematical details of the proofs.

A. The Unit Resource Capacity Region

Our first result determines what rates are achievable when there is no noisy resource—the only resources available are noiseless classical communication, noiseless quantum communication, and noiseless entanglement. We provide a 3-D "unit resource" capacity region that lives in a 3-D space with points (C, Q, E).

Three important protocols relate the three fundamental noiseless resources. These protocols are TP [9], SD [10], and ED [23]. We can express these three protocols as resource inequalities. The resource inequality for TP is

$$2[c \to c] + [qq] \geq [q \to q] \tag{1}$$

where the meaning of the resource inequality is as before—the protocol consumes the resources on the left in order to produce the resource on the right. SD corresponds to the following inequality:

$$[q \to q] + [qq] \geq 2[c \to c] \tag{2}$$

and ED is as follows:

$$[q \to q] \geq [qq]. \tag{3}$$

A sender implements ED by transmitting half of a locally prepared Bell state $|\Phi^+\rangle$ through a noiseless qubit channel. In any tradeoff problem, we have the achievable rate region and the capacity region. The *achievable rate region* is the set of all rate triples that one can achieve with a specific, known protocol. The *capacity region* divides the line between what is physically achievable and what is not—there is no method to achieve any point outside the capacity region. We define it with respect to a given quantum information processing task. In our development below, we consider the achievable rate region and the capacity region of the three unit resources of noiseless classical communication, noiseless quantum communication, and noiseless entanglement.

Definition 1: Let $\tilde{\mathcal{C}}_{\mathrm{U}}$ denote the unit resource achievable rate region. It consists of all the rate triples (C, Q, E) obtainable from linear combinations of the above protocols: TP, SD, and ED.

The development in Section IV demonstrates that the achievable rate region $\tilde{\mathcal{C}}_{\mathrm{U}}$ in the above definition is equivalent to all rate triples satisfying the following inequalities:

$$C + Q + E \leq 0, \quad Q + E \leq 0, \quad C + 2Q \leq 0. \tag{4}$$

Definition 2: The unit resource capacity region $\mathcal{C}_{\mathrm{U}}$ is the closure of the set of all points (C, Q, E) in the C, Q, E space satisfying the following resource inequality:

$$0 \geq C[c \to c] + Q[q \to q] + E[qq]. \tag{5}$$

The above notation may seem confusing at first glance until we establish the convention that a resource with a negative rate implicitly belongs on the left-hand side of the resource inequality.

Theorem 1 below is our first main result, giving the optimal 3-D capacity region for the three unit resources.

Theorem 1: The unit resource capacity region $\mathcal{C}_{\mathrm{U}}$ is equivalent to the unit resource achievable rate region $\tilde{\mathcal{C}}_{\mathrm{U}}$

$$\mathcal{C}_{\mathrm{U}} = \tilde{\mathcal{C}}_{\mathrm{U}}.$$

The complete proof is in Section IV. It involves several proofs by contradiction that apply to each octant of the (C, Q, E) space. It exploits two postulates: 1) ebits alone cannot generate cbits or qubits and 2) cbits alone cannot generate ebits or qubits.

B. Direct-Static Capacity Region

Our second result provides a solution to the scenario when a noisy static resource is available in addition to the three noiseless resources. We determine a 3-D "direct-static" capacity region that gives multiletter formulas for the full tradeoff between the three fundamental noiseless resources.

Definition 3: The direct-static capacity region $\mathcal{C}_{\mathrm{DS}}(\rho^{AB})$ of a noisy bipartite state ρ^{AB} is a 3-D region in the (C, Q, E) space. It is the closure of the set of all points (C, Q, E) satisfying the following resource inequality:

$$\langle \rho^{AB} \rangle \geq C[c \to c] + Q[q \to q] + E[qq]. \tag{6}$$

The rates C, Q, and E can either be negative or positive with the same interpretation as in the previous section.

We first introduce a new protocol that proves to be useful in determining the achievable rate region for the static case. We name this protocol "classically assisted quantum state redistribution."

Lemma 1: The following "classically assisted quantum state redistribution" resource inequality holds:

$$\langle \rho^{AB} \rangle + \frac{1}{2} I(A'; E|E'X)_\sigma [q \to q] + I(X; E|B)_\sigma [c \to c] \geq \frac{1}{2} \left(I(A'; B|X)_\sigma - I(A'; E'|X)_\sigma \right) [qq] \tag{7}$$

for a static resource ρ^{AB} and for any remote instrument $\mathcal{T}^{A\rightarrow A'X}$. In the above resource inequality, the state $\sigma^{XA'BEE'}$ is defined by

$$\sigma^{XA'BEE'} \equiv \tilde{\mathcal{T}}(\psi^{ABE}) \tag{8}$$

where $|\psi\rangle\langle\psi|^{ABE}$ is some purification of ρ^{AB} and $\tilde{\mathcal{T}}^{A\rightarrow A'E'X}$ is an extension of $\mathcal{T}^{A\rightarrow A'X}$.

The above quantities $I(A';E|E'X)_\sigma$, $I(X;E|B)_\sigma$, and $I(A';B|X)_\sigma - I(A';E'|X)_\sigma$ are entropic quantities that are taken with respect to the state $\sigma^{XA'BEE'}$. These quantities give the rates of resource consumption or generation in the above protocol. We refer the reader to [25] and [31] for definitions of the above entropic quantities and the definition of a "quantum instrument." The above protocol generalizes the mother protocol [25], noisy TP [25], noisy SD [25], the entanglement distillation protocol [16], and the grandmother protocol [25].

Definition 4: The classically assisted state redistribution "one-shot" achievable rate region $\tilde{\mathcal{C}}^{(1)}_{\mathrm{CASR}}(\rho^{AB})$ is as follows:

$$\tilde{\mathcal{C}}^{(1)}_{\mathrm{CASR}}(\rho^{AB}) \equiv \bigcup_{\tilde{\mathcal{T}}} \begin{pmatrix} -I(X;E|B)_\sigma, -\frac{1}{2}I(A';E|E'X)_\sigma, \\ \frac{1}{2}(I(A';B|X)_\sigma - I(A';E'|X)_\sigma) \end{pmatrix}$$

where σ is defined as above and the union is over all instruments $\tilde{\mathcal{T}}$. The classically assisted state redistribution achievable rate region $\tilde{\mathcal{C}}_{\mathrm{CASR}}(\rho^{AB})$ is the following multiletter regularization of the one-shot region:

$$\tilde{\mathcal{C}}_{\mathrm{CASR}}(\rho^{AB}) \equiv \overline{\bigcup_{k=1}^{\infty} \frac{1}{k}\tilde{\mathcal{C}}^{(1)}_{\mathrm{CASR}}((\rho^{AB})^{\otimes k})}. \tag{9}$$

Below we state our second main result, the direct-static capacity theorem.

Theorem 2: The direct-static capacity region $\mathcal{C}_{\mathrm{DS}}(\rho^{AB})$ is equivalent to the direct-static achievable rate region $\tilde{\mathcal{C}}_{DS}(\rho^{AB})$

$$\mathcal{C}_{\mathrm{DS}}(\rho^{AB}) = \tilde{\mathcal{C}}_{\mathrm{DS}}(\rho^{AB}).$$

The direct-static achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DS}}(\rho^{AB})$ is the set addition of the classically assisted state redistribution achievable rate region $\tilde{\mathcal{C}}_{\mathrm{CASR}}$ and the unit resource achievable rate region $\tilde{\mathcal{C}}_{\mathrm{U}}$

$$\tilde{\mathcal{C}}_{\mathrm{DS}}(\rho^{AB}) \equiv \tilde{\mathcal{C}}_{\mathrm{CASR}}(\rho^{AB}) + \tilde{\mathcal{C}}_{\mathrm{U}}. \tag{10}$$

The complete proof is in Section VI. The meaning of the theorem is that it is possible to obtain all achievable points in the direct-static capacity region by combining only four protocols: classically assisted state redistribution, SD, TP, and ED.

The "one-shot," "one-instrument" direct-static achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DS},\tilde{\mathcal{T}}}(\rho^{AB})$ admits the following alternative representation in terms of a set of information-theoretic inequalities:

$$\begin{aligned} C + 2Q &\leq -I(X;E|B)_\sigma - I(A';E|E'X)_\sigma \\ Q + E &\leq I(A'\rangle BX)_\sigma \\ C + Q + E &\leq I(A'\rangle BX)_\sigma - I(X;E|B)_\sigma \end{aligned}$$

where σ is as in (8). This region is the translation of the unit resource achievable region to the classically assisted state redistribution protocol. The regularization of this region and the union over all instruments gives the full direct-static region.

C. Direct-Dynamic Tradeoff

Our third result is a 3-D "direct-dynamic" capacity region that gives the full tradeoff between the three fundamental noiseless resources when a noisy dynamic resource is available.

Definition 5: The direct-dynamic capacity region $\mathcal{C}_{\mathrm{DD}}(\mathcal{N})$ of a noisy channel $\mathcal{N}^{A'\rightarrow B}$ is a 3-D region in the (C,Q,E) space defined by the closure of the set of all points (C,Q,E) satisfying the following resource inequality:

$$\langle\mathcal{N}\rangle \geq C[c\rightarrow c] + Q[q\rightarrow q] + E[qq]. \tag{11}$$

We first recall a few theorems concerning the classically enhanced father protocol [31] because this protocol proves useful in determining the achievable rate region for the dynamic case. Briefly, the classically enhanced father protocol gives a way to transmit classical and quantum information over an entanglement-assisted quantum channel.

Lemma 2: The following classically enhanced father resource inequality holds:

$$\begin{aligned} \langle\mathcal{N}\rangle &+ \frac{1}{2}I(A;E|X)_\sigma[qq] \\ &\geq \frac{1}{2}I(A;B|X)_\sigma[q\rightarrow q] + I(X;B)_\sigma[c\rightarrow c] \end{aligned} \tag{12}$$

for a noisy dynamic resource $\mathcal{N}^{A'\rightarrow B}$. In the above resource inequality, the state σ^{XABE} is defined as follows:

$$\sigma^{XABE} \equiv \sum_x p(x)\,|x\rangle\langle x|^X \otimes U_{\mathcal{N}}^{A'\rightarrow BE}(\psi_x^{AA'}) \tag{13}$$

where the states $\psi_x^{AA'}$ are pure and $U_{\mathcal{N}}^{A'\rightarrow BE}$ is an isometric extension of $\mathcal{N}$.

The classically enhanced father protocol generalizes the father protocol [25], classically enhanced quantum communication [21], entanglement-assisted classical communication [24], classical communication [4], [5], and quantum communication [6]–[8].

Definition 6: The "one-shot" classically enhanced father achievable rate region $\tilde{\mathcal{C}}^{(1)}_{\mathrm{CEF}}(\mathcal{N})$ is as follows:

$$\tilde{\mathcal{C}}^{(1)}_{\mathrm{CEF}}(\mathcal{N}) \equiv \bigcup_{\sigma}\left(I(X,B)_\sigma, \frac{1}{2}I(A;B|X)_\sigma, -\frac{1}{2}I(A;E|X)_\sigma\right)$$

where σ is defined in (13). The classically enhanced father achievable rate region $\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N})$ is the following multiletter regularization of the one-shot region:

$$\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) = \overline{\bigcup_{k=1}^{\infty}\frac{1}{k}\tilde{\mathcal{C}}^{(1)}_{\mathrm{CEF}}(\mathcal{N}^{\otimes k})}.$$

We now state our third main result, the direct-dynamic capacity theorem.

Theorem 3: The direct-dynamic capacity region $\mathcal{C}_{\rm DD}(\mathcal{N})$ is equivalent to the direct-dynamic achievable rate region $\tilde{\mathcal{C}}_{\rm DD}(\mathcal{N})$

$$\mathcal{C}_{\rm DD}(\mathcal{N}) = \tilde{\mathcal{C}}_{\rm DD}(\mathcal{N}).$$

The direct-dynamic achievable rate region $\tilde{\mathcal{C}}_{\rm DD}(\mathcal{N})$ is the set addition of the classically enhanced father achievable rate region and the unit resource achievable rate region

$$\tilde{\mathcal{C}}_{\rm DD}(\mathcal{N}) \equiv \tilde{\mathcal{C}}_{\rm CEF}(\mathcal{N}) + \tilde{\mathcal{C}}_{\rm U}. \tag{14}$$

The complete proof is in Section VII. The meaning of the theorem is that it is possible to obtain all achievable points in the direct-dynamic capacity region by combining only four protocols: the classically enhanced father protocol, SD, TP, and ED.

The "one-shot," "one-state" direct-dynamic achievable rate region $\tilde{\mathcal{C}}^{(1)}_{{\rm DD},\sigma}(\mathcal{N})$ admits the following alternative representation in terms of a set of information-theoretic inequalities:

$$\begin{aligned} C + 2Q &\leq I(AX;B)_\sigma \\ Q + E &\leq I(A\rangle BX)_\sigma \\ C + Q + E &\leq I(X;B)_\sigma + I(A\rangle BX)_\sigma \end{aligned}$$

where σ is as in (13). This region is the translation of the unit resource achievable region to the classically enhanced father protocol. The regularization of this region and the union over all states gives the full direct-dynamic region.

IV. The Triple Tradeoff Between Unit Resources

We now consider what rates are achievable when there is no noisy resource—the only resources available are noiseless classical communication, noiseless quantum communication, and noiseless entanglement. We prove our first main result: Theorem 1. Recall that this theorem states that the 3-D "unit resource" achievable rate region, involving the three fundamental noiseless resources, is equivalent to the unit resource capacity region.

In the unit resource capacity theorem, we exploit the following geometric objects that lie in the (C,Q,E) space.

1) TP is the point $(-2,1,-1)$. The "line of TP" $L_{\rm TP}$ is the following set of points:

$$L_{\rm TP} \equiv \{\alpha(-2,1,-1) : \alpha \geq 0\}. \tag{15}$$

2) SD is the point $(2,-1,-1)$. The "line of SD" $L_{\rm SD}$ is the following set of points:

$$L_{\rm SD} \equiv \{\beta(2,-1,-1) : \beta \geq 0\}. \tag{16}$$

3) ED is the point $(0,-1,1)$. The "line of ED" $L_{\rm ED}$ is the following set of points:

$$L_{\rm ED} \equiv \{\gamma(0,-1,1) : \gamma \geq 0\}. \tag{17}$$

Let $\tilde{\mathcal{C}}_{\rm U}$ denote the unit resource achievable rate region. It consists of all linear combinations of the above protocols

$$\tilde{\mathcal{C}}_{\rm U} \equiv L_{\rm TP} + L_{\rm SD} + L_{\rm ED}. \tag{18}$$

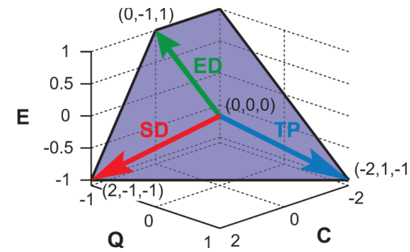


Fig. 1. The unit resource achievable region $\tilde{\mathcal{C}}_{\rm U}$. The blue, red, and green lines are the respective lines of TP, SD, and ED defined in (15)–(17). These three lines and their set addition bound the unit resource capacity region.

The following matrix equation gives all achievable triples (C,Q,E) in $\tilde{\mathcal{C}}_{\rm U}$:

$$\begin{bmatrix} C \\ Q \\ E \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

where $\alpha, \beta, \gamma \geq 0$. We can rewrite the above equation with its matrix inverse

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -1 & 0 \end{bmatrix} \begin{bmatrix} C \\ Q \\ E \end{bmatrix}$$

in order to express the coefficients α, β, and γ as a function of the rate triples (C,Q,E). The restriction of nonnegativity of α, β, and γ gives the following restriction on the achievable rate triples (C,Q,E):

$$C + Q + E \leq 0 \tag{19}$$
$$Q + E \leq 0 \tag{20}$$
$$C + 2Q \leq 0. \tag{21}$$

The above result implies that the achievable rate region $\tilde{\mathcal{C}}_{\rm U}$ in (18) is equivalent to all rate triples satisfying (19)–(21). Fig. 1 displays the full unit resource achievable rate region.

Proving Theorem 1 involves two steps, traditionally called the *direct coding theorem* and the *converse theorem* . For this case, the *direct coding theorem* establishes that the achievable rate region $\tilde{\mathcal{C}}_{\rm U}$ belongs to the capacity region $\mathcal{C}_{\rm U}$

$$\tilde{\mathcal{C}}_{\rm U} \subseteq \mathcal{C}_{\rm U}.$$

The *converse theorem*, on the other hand, establishes the other inclusion

$$\mathcal{C}_{\rm U} \subseteq \tilde{\mathcal{C}}_{\rm U}.$$

A. *Proof of the Direct Coding Theorem*

The result of the direct coding theorem, that $\tilde{\mathcal{C}}_{\rm U} \subseteq \mathcal{C}_{\rm U}$, is immediate from the definition in (18) of the unit resource achievable rate region $\tilde{\mathcal{C}}_{\rm U}$, the definition in (5) of the unit resource capacity region $\mathcal{C}_{\rm U}$, and the theory of resource inequalities in [23] and [25].

B. *Proof of the Converse Theorem*

We employ the definition of $\tilde{\mathcal{C}}_{\mathrm{U}}$ in (18) and consider the eight octants of the (C, Q, E) space individually in order to prove the converse theorem (that $\mathcal{C}_{\mathrm{U}} \subseteq \tilde{\mathcal{C}}_{\mathrm{U}}$). Let $(\pm, \pm, \pm)$ denote labels for the eight different octants.

It is possible to demonstrate the optimality of each of these three protocols individually with a contradiction argument (e.g., the contradiction argument for TP is in [9]). However, in the converse proof of Theorem 1, we show that a mixed strategy combining these three noiseless protocols is optimal.

We accept the following two postulates and exploit them in order to prove the converse.

1) Entanglement alone cannot generate classical communication or quantum communication or both.
2) Classical communication alone cannot generate entanglement or quantum communication or both.

$(+,+,+)$. This octant of $\mathcal{C}_{\mathrm{U}}$ is empty because a sender and a receiver require some resources to implement classical communication, quantum communication, and entanglement. (They cannot generate a noiseless resource from nothing.)

$(+,+,-)$. This octant of $\mathcal{C}_{\mathrm{U}}$ is empty because entanglement alone cannot generate either classical communication or quantum communication or both.

$(+,-,+)$. The task for this octant is to generate a noiseless classical channel of C bits and E ebits of entanglement using $|Q|$ qubits of quantum communication. We thus consider all points of the form (C, Q, E) where $C \geq 0$, $Q \leq 0$, and $E \geq 0$. It suffices to prove the following inequality:

$$C + E \leq |Q| \tag{22}$$

because combining (22) with $C \geq 0$ and $E \geq 0$ implies (19)–(21). The achievability of $(C, -|Q|, E)$ implies the achievability of the point $(C + 2E, -|Q| - E, 0)$, because we can consume all of the entanglement with SD (2). This new point implies that there is a protocol that consumes $|Q| + E$ noiseless qubit channels to send $C + 2E$ classical bits. The following bound then applies:

$$C + 2E \leq |Q| + E$$

because the Holevo bound [35] states that we can send only one classical bit per qubit. The bound in (22) then follows.

$(+,-,-)$. The task for this octant is to simulate a classical channel of size C bits using $|Q|$ qubits of quantum communication and $|E|$ ebits of entanglement. We consider all points of the form (C, Q, E) where $C \geq 0$, $Q \leq 0$, and $E \leq 0$. It suffices to prove the following inequalities:

$$C \leq 2|Q| \tag{23}$$
$$C \leq |Q| + |E| \tag{24}$$

because combining (23) and (24) with $C \geq 0$ implies (19)–(21). The achievability of $(C, -|Q|, -|E|)$ implies the achievability of $(0, -|Q| + C/2, -|E| - C/2)$, because we can consume all of the classical communication with TP (1). The following bound applies (quantum communication cannot be positive):

$$-|Q| + \frac{C}{2} \leq 0$$

because entanglement alone cannot generate quantum communication. The bound in (23) then follows from the above bound. The achievability of $(C, -|Q|, -|E|)$ implies the achievability of $(C, -|Q| - |E|, 0)$ because we can consume an extra $|E|$ qubit channels with ED (3). The bound in (24) then applies by the same Holevo bound argument as in the previous octant.

$(-,+,+)$. This octant of $\mathcal{C}_{\mathrm{U}}$ is empty because classical communication alone cannot generate either quantum communication or entanglement or both.

$(-,+,-)$. The task for this octant is to simulate a quantum channel of size Q qubits using $|E|$ ebits of entanglement and $|C|$ bits of classical communication. We consider all points of the form (C, Q, E) where $C \leq 0$, $Q \geq 0$, and $E \leq 0$. It suffices to prove the following inequalities:

$$Q \leq |E| \tag{25}$$
$$2Q \leq |C| \tag{26}$$

because combining them with $C \leq 0$ implies (19)–(21). The achievability of the point $(-|C|, Q, -|E|)$ implies the achievability of the point $(-|C|, 0, Q - |E|)$, because we can consume all of the quantum communication for ED (3). The following bound applies (entanglement cannot be positive):

$$Q - |E| \leq 0$$

because classical communication alone cannot generate entanglement. The bound in (25) follows from the above bound. The achievability of the point $(-|C|, Q, -|E|)$ implies the achievability of the point $(-|C| + 2Q, 0, -Q - |E|)$, because we can consume all of the quantum communication for SD (2). The following bound applies (classical communication cannot be positive):

$$-|C| + 2Q \leq 0$$

because entanglement alone cannot create classical communication. The bound in (26) follows from the above bound.

$(-,-,+)$. The task for this octant is to create E ebits of entanglement using $|Q|$ qubits of quantum communication and $|C|$ bits of classical communication. We consider all points of the form (C, Q, E) where $C \leq 0$, $Q \leq 0$, and $E \geq 0$. It suffices to prove the following inequality:

$$E \leq |Q| \tag{27}$$

because combining it with $Q \leq 0$ and $C \leq 0$ implies (19)–(21). The achievability of $(-|C|, -|Q|, E)$ implies the achievability of $(-|C| - 2E, -|Q| + E, 0)$, because we can consume all of the entanglement with TP (1). The following bound applies (quantum communication cannot be positive):

$$-|Q| + E \leq 0$$

because classical communication alone cannot generate quantum communication. The bound in (27) follows from the above bound.

$(-,-,-)$. $\tilde{\mathcal{C}}_{\text{U}}$ completely contains this octant.

V. Classically Assisted Quantum State Redistribution

Before discussing the direct-static tradeoff, we overview the *classically assisted quantum state redistribution protocol* introduced in Section III. This protocol generates entanglement with the help of classical communication, quantum communication, and a noisy bipartite state. It employs techniques from Winter's instrument compression theorem [47], Devetak and Winter's "classical compression with quantum side information" theorem [20], and the quantum state redistribution protocol [29], [30], [48]–[50]. Thus, we do not give a full, detailed proof of Lemma 1 (the coding theorem for the protocol), but instead resort to the resource inequality framework [25] for a simple proof of the coding theorem. A full proof of this protocol will appear elsewhere. For now, our simple expository proof of Lemma 1 appears below.

Proof of Lemma 1: Consider a state ρ^{AB} shared between Alice and Bob. Let ψ^{ABE} denote the purification of this state. A remote instrument $\mathcal{T}^{A\rightarrow A'X_BE'}$ acting on this state produces the state $\sigma^{X_BA'BEE'}$ where

$$\sigma^{X_BA'BEE'} \equiv \mathcal{T}^{A\rightarrow A'X_BE'}\left(\psi^{ABE}\right).$$

There exists a protocol, instrument compression with quantum side information (ICQSI), that exploits the techniques from [20] and [47]. It implements the following resource inequality:

$$\left\langle\rho^{AB}\right\rangle + I(X_B;E|B)_\sigma\,[c\rightarrow c] + H(X_B|BE)_\sigma\,[cc] \geq \langle\overline{\Delta}^{X\rightarrow X_AX_B}\circ\mathcal{T}:\rho^A\rangle$$

where $[cc]$ represents the resource of one bit of common randomness and $\overline{\Delta}^{X\rightarrow X_AX_B}$ denotes a classical channel that transmits classical information to Alice and Bob. ICQSI is similar to Winter's instrument compression protocol in the sense that Alice and Bob are exploiting classical communication and common randomness to simulate the action of a quantum instrument, but the difference is that Alice does not need to send as much classical information as required in Winter's instrument compression protocol. Bob can exploit his quantum side information to learn something about the classical information that Alice is transmitting. The static version of the HSW coding theorem from [20] shows that Bob can learn $nI(X_B;B)$ bits about the classical information that Alice is transmitting, so that she does not have to transmit the full $nI(X_B;EB)_\sigma$ bits required by the instrument compression protocol, but instead transmits only $nI(X_B;E|B)_\sigma$ classical bits. It then follows that

$$\left\langle\rho^{AB}\right\rangle + I(X_B;E|B)_\sigma\,[c\rightarrow c] + H(X_B|BE)_\sigma\,[cc] \geq \langle\overline{\Delta}^{X\rightarrow X_AX_B}(\sigma^{XA'BEE'})\rangle$$

because simulating the quantum instrument is the same as actually performing it in the asymptotic limit.

We now apply the quantum state redistribution protocol discussed in [29], [30], and [48]–[50]. This protocol is useful here because it makes efficient use of both systems A' and E' that Alice possesses after the action of the instrument. We specifically apply the "reversed" version of state redistribution outlined on the right-hand side of [48, Fig. 3], with the substitutions $R\leftrightarrow E$, $B\leftrightarrow E'$, $C\leftrightarrow A'$, and $A\leftrightarrow B$. Finally, we can also apply [25, Th. 4.12], which shows how convex combinations of static resources are related to conditioning on classical variables, to get the following resource inequality:

$$\langle\overline{\Delta}^{X\rightarrow X_AX_B}(\sigma^{X_BA'BEE'})\rangle + \frac{1}{2}I(A';E|E'X_B)_\sigma\,[q\rightarrow q] \geq \frac{1}{2}\left(I(A';B|X_B)_\sigma - I(A';E|X_B)_\sigma\right)[qq].$$

Combining the above two resource inequalities gives the following resource inequality:

$$\begin{aligned}\left\langle\rho^{AB}\right\rangle &+ I(X_B;E|B)_\sigma\,[c\rightarrow c] + H(X_B|BE)_\sigma\,[cc]\\ &+ \frac{1}{2}I(A';E|E'X_B)_\sigma\,[q\rightarrow q]\\ &\geq \frac{1}{2}\left(I(A';B|X_B)_\sigma - I(A';E'|X_B)_\sigma\right)[qq].\end{aligned}$$

We can then derandomize the protocol and eliminate the common randomness via [25, Corollary 4.8] because the output resource is pure. The result is the following resource inequality:

$$\left\langle\rho^{AB}\right\rangle + I(X_B;E|B)_\sigma\,[c\rightarrow c] + \frac{1}{2}I(A';E|E'X_B)_\sigma\,[q\rightarrow q] \geq \frac{1}{2}\left(I(A';B|X_B)_\sigma - I(A';E'|X_B)_\sigma\right)[qq].$$

□

We obtain the classically assisted state redistribution "one-shot" achievable rate region $\tilde{\mathcal{C}}^{(1)}_{\text{CASR}}(\rho^{AB})$ and its regularization $\tilde{\mathcal{C}}_{\text{CASR}}(\rho^{AB})$ in Definition 4 by performing the protocol with all possible instruments and taking the union over the resulting achievable rates.

Corollary 4: The following "grandmother" protocol of [25] results from combining the classically assisted state redistribution resource inequality with ED of $\frac{n}{2}I(A';E'|X_B)_\sigma$ extra qubits of quantum communication:

$$\left\langle\rho^{AB}\right\rangle + I(X_B;E|B)_\sigma\,[c\rightarrow c] + \frac{1}{2}I(A';EE'|X_B)_\sigma\,[q\rightarrow q] \geq \frac{1}{2}I(A';B|X_B)_\sigma\,[qq].$$

The above resource inequality requires slightly less classical communication than the grandmother protocol from [25].

VI. Direct-Static Tradeoff

In this section, we prove Theorem 2, the direct-static capacity theorem. Recall that this theorem determines what rates are achievable when a sender and a receiver consume a noisy static resource. The parties additionally consume or generate noiseless classical communication, noiseless quantum communication, and noiseless entanglement. The theorem determines the 3-D "direct-static" capacity region that gives the full tradeoff between the three fundamental noiseless resources when a noisy static resource is available.

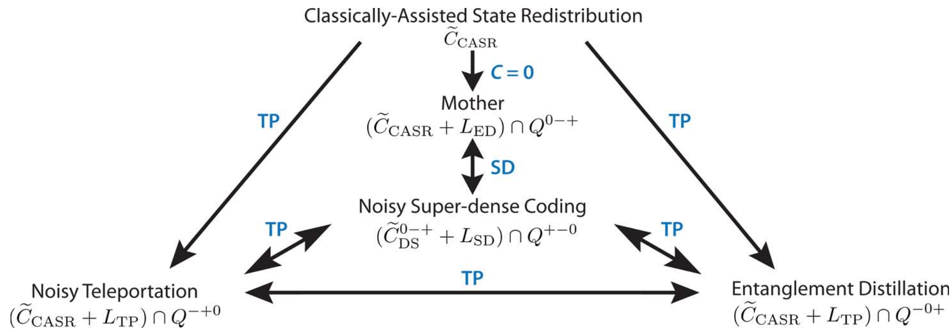


Fig. 2. Relations between the achievable rate regions for the static case. Any bidirectional arrow represents a bijection between two achievable rate regions. Any one-way arrow represents an injection from one achievable rate region to another.

A. *Proof of the Direct Coding Theorem*

The direct coding theorem is the statement that the direct-static capacity region contains the direct-static achievable rate region

$$\tilde{\mathcal{C}}_{\mathrm{DS}}(\rho^{AB}) \subseteq \mathcal{C}_{\mathrm{DS}}(\rho^{AB}).$$

It follows directly from combining the classically assisted state redistribution resource inequality (7) with TP (1), SD (2), and ED (3) and considering the definition of the direct-static achievable rate region in (10) and the definition of the direct-static capacity region in (6).

B. *Proof of the Converse Theorem*

The converse theorem is the statement that the direct-static achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DS}}(\rho^{AB})$ contains the direct-static capacity region $\mathcal{C}_{\mathrm{DS}}(\rho^{AB})$

$$\mathcal{C}_{\mathrm{DS}}(\rho^{AB}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}(\rho^{AB}).$$

In order to prove it, we consider one octant of the (C,Q,E) space at a time and use the notation from Section II. We omit writing ρ^{AB} in what follows and instead write ρ to denote the noisy bipartite state ρ^{AB}.

The converse proof exploits relations among the previously known capacity regions corresponding to the following operational tasks: classically assisted state redistribution, the mother protocol [23], [25], noisy SD [19], [25], noisy TP [23], [25], and entanglement distillation [25]. We illustrate the relation between these protocols in Fig. 2.

Quantum-communication-assisted entanglement distillation consumes a noisy static resource and noiseless quantum communication to generate noiseless entanglement. The mother protocol is a particular protocol for such a task. The mother's achievable rate region $\tilde{\mathcal{C}}^{0-+}(\rho)$ and the capacity region $\mathcal{C}^{0-+}(\rho)$ lie in the Q^{0-+} quadrant of the (C,Q,E) space

$$\tilde{\mathcal{C}}^{0-+}_{\mathrm{DS}}(\rho) = \tilde{\mathcal{C}}_{\mathrm{DS}}(\rho) \cap Q^{0-+} \tag{28}$$
$$\mathcal{C}^{0-+}_{\mathrm{DS}}(\rho) = \mathcal{C}_{\mathrm{DS}}(\rho) \cap Q^{0-+}. \tag{29}$$

The above relations establish that these respective regions are special cases of the direct-static achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DS}}(\rho)$ and the direct-static capacity region $\mathcal{C}_{\mathrm{DS}}(\rho)$. The quantum-communication-assisted entanglement distillation capacity theorem states that the mother's achievable rate region $\tilde{\mathcal{C}}^{0-+}_{\mathrm{DS}}(\rho)$ is the same as the capacity region

$$\tilde{\mathcal{C}}^{0-+}_{\mathrm{DS}}(\rho) = \mathcal{C}^{0-+}_{\mathrm{DS}}(\rho). \tag{30}$$

The mother's achievable rate region $\tilde{\mathcal{C}}^{0-+}_{\mathrm{DS}}(\rho)$ is a special case of the classically assisted state redistribution achievable rate region $\tilde{\mathcal{C}}_{\mathrm{CASR}}(\rho)$ (combined with ED) where there is no consumption of classical communication

$$\tilde{\mathcal{C}}^{0-+}_{\mathrm{DS}}(\rho) = (\tilde{\mathcal{C}}_{\mathrm{CASR}}(\rho) + L_{\mathrm{ED}}) \cap Q^{0-+}. \tag{31}$$

Quantum-communication-assisted classical communication consumes a noisy static resource and noiseless quantum communication to generate noiseless classical communication. The noisy SD protocol is a particular protocol for such a task. The noisy SD achievable rate region $\tilde{\mathcal{C}}^{+-0}_{\mathrm{DS}}(\rho)$ and the capacity region $\mathcal{C}^{+-0}_{\mathrm{DS}}(\rho)$ lie in the Q^{+-0} quadrant of the (C,Q,E) space

$$\tilde{\mathcal{C}}^{+-0}_{\mathrm{DS}}(\rho) = \tilde{\mathcal{C}}_{\mathrm{DS}}(\rho) \cap Q^{+-0} \tag{32}$$
$$\mathcal{C}^{+-0}_{\mathrm{DS}}(\rho) = \mathcal{C}_{\mathrm{DS}}(\rho) \cap Q^{+-0}. \tag{33}$$

The above relations establish that these regions are special cases of the direct-static achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DS}}(\rho)$ and the direct-static capacity region $\mathcal{C}_{\mathrm{DS}}(\rho)$. The quantum-communication-assisted classical communication capacity theorem states that noisy SD's achievable rate region is equivalent to the capacity region

$$\tilde{\mathcal{C}}^{+-0}_{\mathrm{DS}}(\rho) = \mathcal{C}^{+-0}_{\mathrm{DS}}(\rho). \tag{34}$$

Its achievable rate region $\tilde{\mathcal{C}}^{+-0}_{\mathrm{DS}}(\rho)$ is obtainable from the mother's achievable rate region $\tilde{\mathcal{C}}^{0-+}_{\mathrm{DS}}(\rho)$ by combining it with SD and keeping the points with zero entanglement

$$\begin{aligned}\tilde{\mathcal{C}}^{+-0}_{\mathrm{DS}}(\rho) &= \left(\tilde{\mathcal{C}}^{0-+}_{\mathrm{DS}}(\rho) + L_{\mathrm{SD}}\right) \cap Q^{+-0} \\ &= \left(((\tilde{\mathcal{C}}_{\mathrm{CASR}}(\rho) + L_{\mathrm{ED}}) \cap Q^{0-+}) + L_{\mathrm{SD}}\right) \\ &\quad \cap Q^{+-0}\end{aligned} \tag{35}$$

where the second equality comes from (31).

Classical-communication-assisted quantum communication consumes a noisy static resource and noiseless classical communication to generate noiseless quantum communication. The noisy TP protocol is a particular protocol for such a task.

Its achievable rate region $\tilde{\mathcal{C}}_{\rm DS}^{-+0}(\rho)$ and the capacity region $\mathcal{C}_{\rm DS}^{-+0}(\rho)$ lie in the Q^{-+0} quadrant of the (C,Q,E) space

$$\tilde{\mathcal{C}}_{\rm DS}^{-+0}(\rho) = \tilde{\mathcal{C}}_{\rm DS}(\rho) \cap Q^{-+0} \tag{36}$$
$$\mathcal{C}_{\rm DS}^{-+0}(\rho) = \mathcal{C}_{\rm DS}(\rho) \cap Q^{-+0}. \tag{37}$$

The above relations establish that these respective regions are special cases of the direct-static achievable rate region $\tilde{\mathcal{C}}_{\rm DS}(\rho)$ and the direct-static capacity region $\mathcal{C}_{\rm DS}(\rho)$. The classical-communication-assisted quantum communication capacity theorem states that the achievable rate region is equivalent to the capacity region

$$\tilde{\mathcal{C}}_{\rm DS}^{-+0}(\rho) = \mathcal{C}_{\rm DS}^{-+0}(\rho). \tag{38}$$

The achievable rate region $\tilde{\mathcal{C}}_{\rm DS}^{-+0}(\rho)$ is obtainable from the classically assisted state redistribution's achievable rate region $\tilde{\mathcal{C}}_{\rm CASR}(\rho)$ by combining it with TP and keeping the points with zero entanglement

$$\tilde{\mathcal{C}}_{\rm DS}^{-+0}(\rho) = (\tilde{\mathcal{C}}_{\rm CASR}(\rho) + L_{\rm TP}) \cap Q^{-+0}. \tag{39}$$

Classically assisted entanglement distillation consumes a noisy static resource and noiseless classical communication to generate noiseless entanglement. The BDSW96 entanglement distillation protocol is a particular protocol for such a task [16]. The achievable rate region $\tilde{\mathcal{C}}_{\rm DS}^{-0+}(\rho)$ and the capacity region $\mathcal{C}_{\rm DS}^{-0+}(\rho)$ lie in the Q^{-0+} quadrant of the (C,Q,E) space

$$\tilde{\mathcal{C}}_{\rm DS}^{-0+}(\rho) = \tilde{\mathcal{C}}_{\rm DS}(\rho) \cap Q^{-0+} \tag{40}$$
$$\mathcal{C}_{\rm DS}^{-0+}(\rho) = \mathcal{C}_{\rm DS}(\rho) \cap Q^{-0+}. \tag{41}$$

The above relations show that these respective regions are special cases of the direct-static achievable rate region $\tilde{\mathcal{C}}_{\rm DS}(\rho)$ and the direct-static capacity region $\mathcal{C}_{\rm DS}(\rho)$. The entanglement distillation capacity theorem states that the achievable rate region is equivalent to the capacity region

$$\tilde{\mathcal{C}}_{\rm DS}^{-0+}(\rho) = \mathcal{C}_{\rm DS}^{-0+}(\rho). \tag{42}$$

Its achievable rate region $\tilde{\mathcal{C}}_{\rm DS}^{-0+}(\rho)$ is obtainable from the classically assisted state redistribution's achievable rate region $\tilde{\mathcal{C}}_{\rm CASR}(\rho)$ by combining it with TP and keeping the points with zero quantum communication

$$\tilde{\mathcal{C}}_{\rm DS}^{-0+}(\rho) = (\tilde{\mathcal{C}}_{\rm CASR}(\rho) + L_{\rm TP}) \cap Q^{-0+}. \tag{43}$$

1) Entanglement-Generating Octants: We first consider the four octants with corresponding protocols that generate entanglement, i.e., those of the form $(\pm,\pm,+)$. The proof of one octant is trivial and the full proof of another appears in Appendix I. The proofs of the remaining two octants are similar to each other—they follow by consuming all of the entanglement in those octants and resorting to the previously established converse theorems for 2-D quadrants. Geometrically, the technique is to project all points in an octant into a quadrant with a known capacity theorem and exploit the converse proof of the corresponding quadrant.

$(+,+,+)$. This octant is empty because a noisy static resource alone cannot generate a dynamic resource.

$(-,-,+)$. The full proof of the converse for this octant appears in Appendix I. There, we obtain the following bounds on the one-shot, one-instrument capacity region

$$\begin{aligned} nE &\leq I(A'\rangle B^n X)_\sigma + |nQ| \\ n(|C| + 2|Q|) &\geq I(X;E^n|B^n)_\sigma + I(A';E^n|E'X)_\sigma \\ nE &\leq n(|C|+|Q|) + I(A'\rangle B^n X)_\sigma \\ &\quad - I(X;E^n|B^n)_\sigma \end{aligned}$$

where the entropies are with respect to a regularized version of the state in (8). The above set of inequalities represents a translation of the unit resource capacity region to the achievable rate triple of the regularized classically assisted state redistribution protocol. The result is that the capacity region for this octant is within an achievable rate region consisting of all rates achieved by the regularized classically assisted state redistribution protocol combined with the unit protocols

$$\mathcal{C}_{\rm DS}^{--+}(\rho) \subseteq \tilde{\mathcal{C}}_{\rm CASR}(\rho) + \tilde{\mathcal{C}}_{\rm U}.$$

$(+,-,+)$. This octant exploits the projection technique with SD. Let

$$\mathcal{C}_{\rm DS}^{+-+}(\rho) \equiv \mathcal{C}_{\rm DS}(\rho) \cap O^{+-+}$$

and recall the definition of $\mathcal{C}_{\rm DS}^{+-0}(\rho)$ in (33). We exploit the line of superdense coding $L_{\rm SD}$ as defined in (16). Define the following maps:

$$f : S \rightarrow (S + L_{\rm SD}) \cap Q^{+-0}$$
$$\hat{f} : S \rightarrow (S - L_{\rm SD}) \cap O^{+-+}.$$

The map f translates the set S in the SD direction and keeps the points that lie on the Q^{+-0} quadrant. The map $\hat{f}$, in a sense, undoes the effect of f by moving the set S back to the octant O^{+-+}.

The inclusion $\mathcal{C}_{\rm DS}^{+-+}(\rho) \subseteq \hat{f}(f(\mathcal{C}_{\rm DS}^{+-+}(\rho)))$ holds because

$$\begin{aligned} &\mathcal{C}_{\rm DS}^{+-+}(\rho) \\ &\quad = \mathcal{C}_{\rm DS}^{+-+}(\rho) \cap O^{+-+} \\ &\quad \subseteq (((\mathcal{C}_{\rm DS}^{+-+}(\rho) + L_{\rm SD}) \cap Q^{+-0}) - L_{\rm SD}) \cap O^{+-+} \\ &\quad = (f(\mathcal{C}_{\rm DS}^{+-+}(\rho)) - L_{\rm SD}) \cap O^{+-+} \\ &\quad = \hat{f}(f(\mathcal{C}_{\rm DS}^{+-+}(\rho))). \end{aligned} \tag{44}$$

The first set equivalence is obvious from the definition of $\mathcal{C}_{\rm DS}^{+-+}(\rho)$. The first inclusion follows from the following logic. Pick any point $a \equiv (C,Q,E) \in \mathcal{C}_{\rm DS}^{+-+}(\rho) \cap O^{+-+}$ and a particular point $b \equiv (2E,-E,-E) \in L_{\rm SD}$. It follows that $a + b = (C+2E, Q-E, 0) \in (\mathcal{C}_{\rm DS}^{+-+}(\rho) + L_{\rm SD}) \cap Q^{+-0}$. We then pick a point $-b = (-2E,E,E) \in -L_{\rm SD}$. It follows that $a+b-b \in (((\mathcal{C}_{\rm DS}^{+-+}(\rho)+L_{\rm SD}) \cap Q^{+-0}) - L_{\rm SD}) \cap O^{+-+}$ and that $a+b-b = (C,Q,E) = a$. The first inclusion thus holds because every point in $\mathcal{C}_{\rm DS}^{+-+}(\rho) \cap O^{+-+}$ is in $(((\mathcal{C}_{\rm DS}^{+-+}(\rho) + L_{\rm SD}) \cap Q^{+-0}) - L_{\rm SD}) \cap O^{+-+}$. The second set equivalence follows from the definition of f and the third set equivalence follows from the definition of $\hat{f}$.

It is operationally clear that the following inclusion holds:

$$f(\mathcal{C}_{\rm DS}^{+-+}(\rho)) \subseteq \mathcal{C}_{\rm DS}^{+-0}(\rho) \tag{45}$$

because the mapping f converts any achievable point $a \in \mathcal{C}_{\rm DS}^{+-+}(\rho)$ to an achievable point in $\mathcal{C}_{\rm DS}^{+-0}(\rho)$ by consuming all of the entanglement at point a with SD.

The converse proof of quantum-communication-assisted classical communication [25] is useful for us

$$\mathcal{C}_{\rm DS}^{+-0}(\rho) \subseteq \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho). \tag{46}$$

Recall the relation in (35) between the achievable rate region $\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho)$ and the classically assisted state redistribution achievable rate region $\tilde{\mathcal{C}}_{\rm CASR}(\rho)$. The following set inclusion holds:

$$\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho) \subseteq (\tilde{\mathcal{C}}_{\rm CASR}(\rho) + L_{\rm SD}) \cap Q^{+-0} \tag{47}$$

by dropping the intersection with Q^{0-+} in (35).

The inclusion $\hat{f}(\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho)) \subseteq \tilde{\mathcal{C}}_{\rm DS}^{+-+}(\rho)$ holds because

$$\begin{aligned}
&\hat{f}(\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho)) \\
&\quad\subseteq (((\tilde{\mathcal{C}}_{\rm CASR}(\rho) + L_{\rm SD}) \cap Q^{+-0}) - L_{\rm SD}) \cap O^{+-+} \\
&\quad\subseteq ((\tilde{\mathcal{C}}_{\rm CASR}(\rho) + L_{\rm SD}) - L_{\rm SD}) \cap O^{+-+} \\
&\quad= ((\tilde{\mathcal{C}}_{\rm CASR}(\rho) + L_{\rm SD}) \cap O^{+-+}) \\
&\qquad \cup ((\tilde{\mathcal{C}}_{\rm CASR}(\rho) - L_{\rm SD}) \cap O^{+-+}) \\
&\quad\subseteq \tilde{\mathcal{C}}_{\rm DS}^{+-+}(\rho).
\end{aligned} \tag{48}$$

The first inclusion follows from (47). The second inclusion follows by dropping the intersection with Q^{+-0}. The second set equivalence follows because $(\tilde{\mathcal{C}}_{\rm CASR}(\rho) + L_{\rm SD}) - L_{\rm SD} = (\tilde{\mathcal{C}}_{\rm CASR}(\rho) + L_{\rm SD}) \cup (\tilde{\mathcal{C}}_{\rm CASR}(\rho) - L_{\rm SD})$, and the last inclusion follows because $(\tilde{\mathcal{C}}_{\rm CASR}(\rho) - L_{\rm SD}) \cap O^{+-+} = (0,0,0)$.

Putting (44), (45), (46), and (48) together, the inclusion $\mathcal{C}_{\rm DS}^{+-+}(\rho) \subseteq \tilde{\mathcal{C}}_{\rm DS}^{+-+}(\rho)$ holds because

$$\begin{aligned}
\mathcal{C}_{\rm DS}^{+-+}(\rho) &\subseteq \hat{f}(f(\mathcal{C}_{\rm DS}^{+-+}(\rho))) \\
&\subseteq \hat{f}(\mathcal{C}_{\rm DS}^{+-0}(\rho)) \subseteq \hat{f}(\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho)) \\
&\subseteq \tilde{\mathcal{C}}_{\rm DS}^{+-+}(\rho).
\end{aligned}$$

The above inclusion is the statement of the converse theorem for this octant.

The proof for the other entanglement generating octant $(-,+,+)$ follows similarly to the above octant by exploiting the same projection technique with TP. The proof appears in Appendix II.

2) Entanglement-Consuming Octants: We now consider all the octants with corresponding protocols that consume entanglement, i.e., those of the form $(\pm,\pm,-)$. The proofs for two of the octants are trivial and the proofs for the two nontrivial octants each contain an additivity lemma that shows how to relate their converse proofs to the converse proofs of a quadrant.

$(+,+,-)$. This octant is empty because a noisy static resource assisted by noiseless entanglement cannot generate a dynamic resource (the two static resources cannot generate classical communication or quantum communication or both).

$(-,-,-)$. $\tilde{\mathcal{C}}_{\rm DS}$ completely contains this octant.

$(+,-,-)$. Define $\Phi^{|E|}$ to be a state of size $|E|$ ebits. We first need the following lemma.

Lemma 3: The following inclusion holds:

$$\mathcal{C}_{\rm DS}^{+-0}(\rho \otimes \Phi^{|E|}) \subseteq \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho) + \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\Phi^{|E|}).$$

Proof: SD induces a linear bijection $f : \tilde{\mathcal{C}}_{\rm DS}^{0-+}(\rho) \to \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho)$ between the mother achievable region $\tilde{\mathcal{C}}_{\rm DS}^{0-+}(\rho)$ and the noisy dense coding achievable region $\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho)$. The bijection f behaves as follows for every point $(0,Q,E) \in \tilde{\mathcal{C}}_{\rm DS}^{0-+}(\rho)$:

$$f : (0,Q,E) \to (2E, Q-E, 0).$$

The following relation holds:

$$f(\tilde{\mathcal{C}}_{\rm DS}^{0-+}(\rho)) = \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho) \tag{49}$$

because applying dense coding to the mother resource inequality gives noisy dense coding [25]. The inclusion $\mathcal{C}_{\rm DS}^{0-+}(\rho \otimes \Phi^{|E|}) \subseteq f^{-1}(\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho) + \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\Phi^{|E|}))$ holds because

$$\begin{aligned}
\mathcal{C}_{\rm DS}^{0-+}(\rho \otimes \Phi^{|E|}) &= \mathcal{C}_{\rm DS}^{0-+}(\rho) + (0,0,E) \\
&\subseteq \mathcal{C}_{\rm DS}^{0-+}(\rho) + \mathcal{C}_{\rm DS}^{0-+}(\Phi^{|E|}) \\
&= \tilde{\mathcal{C}}_{\rm DS}^{0-+}(\rho) + \tilde{\mathcal{C}}_{\rm DS}^{0-+}(\Phi^{|E|}) \\
&= f^{-1}(\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho)) + f^{-1}(\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\Phi^{|E|})) \\
&= f^{-1}(\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho) + \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\Phi^{|E|})).
\end{aligned}$$

The first set equivalence follows because the capacity region of the noisy resource state ρ combined with a rate E maximally entangled state is equivalent to a translation of the capacity region of the noisy resource state ρ. The first inclusion follows because the capacity region of a rate E maximally entangled state contains the rate triple $(0,0,E)$. The second set equivalence follows from the quantum-communication-assisted entanglement distillation capacity theorem in (30), the third set equivalence from (49), and the last from linearity of the map f. The above inclusion implies the following one:

$$f(\mathcal{C}_{\rm DS}^{0-+}(\rho \otimes \Phi^{|E|})) \subseteq \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho) + \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\Phi^{|E|}).$$

The lemma follows because

$$\begin{aligned}
f(\mathcal{C}_{\rm DS}^{0-+}(\rho \otimes \Phi^{|E|})) &= f(\tilde{\mathcal{C}}_{\rm DS}^{0-+}(\rho \otimes \Phi^{|E|})) \\
&= \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho \otimes \Phi^{|E|}) \\
&= \mathcal{C}_{\rm DS}^{+-0}(\rho \otimes \Phi^{|E|})
\end{aligned}$$

where we apply the capacity theorems in (30) and (34). □

We now begin the converse proof for this octant. Observe that

$$\tilde{\mathcal{C}}_{\rm DS}^{+-0}(\Phi^{|E|}) = \tilde{\mathcal{C}}_{\rm U}^{+-E}. \tag{50}$$

Hence, for all $E \le 0$

$$\mathcal{C}_{\rm DS}^{+-E}(\rho) = \mathcal{C}_{\rm DS}^{+-0}(\rho \otimes \Phi^{|E|}) \subseteq \tilde{\mathcal{C}}_{\rm DS}^{+-0}(\rho) + \tilde{\mathcal{C}}_{\rm U}^{+-E} \tag{51}$$

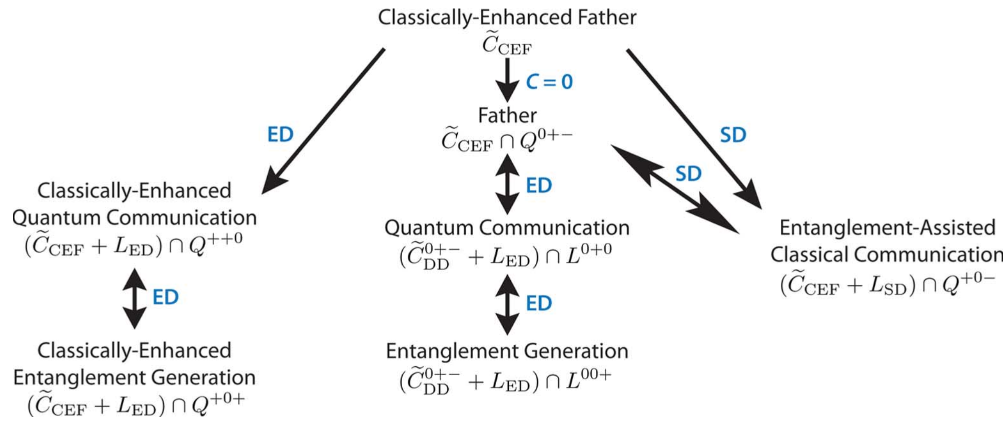


Fig. 3. Relations between the achievable rate regions for the dynamic case. Any bidirectional arrow represents a bijection between two achievable rate regions. Any one-way arrow represents an injection from one achievable rate region to another.

where we apply Lemma 3 and (50). Thus, the inclusion $\mathcal{C}_{\mathrm{DS}}^{+--}(\rho) \subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}^{+--}(\rho)$ holds because

$$\begin{aligned}\mathcal{C}_{\mathrm{DS}}^{+--}(\rho) &= \bigcup_{E\le 0} \mathcal{C}_{\mathrm{DS}}^{+-E}(\rho)\\ &\subseteq \bigcup_{E\le 0} \left(\tilde{\mathcal{C}}_{\mathrm{DS}}^{+-0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{U}}^{+-E}\right)\\ &= (\tilde{\mathcal{C}}_{\mathrm{DS}}^{+-0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{U}}) \cap O^{+--}\\ &\subseteq (\tilde{\mathcal{C}}_{\mathrm{CASR}}(\rho) + \tilde{\mathcal{C}}_{\mathrm{U}}) \cap O^{+--}\\ &= \tilde{\mathcal{C}}_{\mathrm{DS}}^{+--}(\rho).\end{aligned}$$

The first set equivalence holds by definition. The first inclusion follows from (51). The second set equivalence follows because $\bigcup_{E\le 0} \tilde{\mathcal{C}}_{\mathrm{U}}^{+-E} = \tilde{\mathcal{C}}_{\mathrm{U}} \cap O^{+--}$. The second inclusion holds because $\tilde{\mathcal{C}}_{\mathrm{DS}}^{+-0}(\rho)$ is equivalent to noisy dense coding and because the classically assisted state redistribution combined with the unit resource region generates noisy dense coding. The above inclusion $\mathcal{C}_{\mathrm{DS}}^{+--}(\rho) \subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}^{+--}(\rho)$ is the statement of the converse theorem for this octant.

The proof for the other entanglement consuming octant $(-,+,-)$ follows similarly to the above proof and appears in Appendix III.

VII. Direct-Dynamic Tradeoff

In this section, we prove Theorem 3, the direct-dynamic capacity theorem. Recall that this theorem determines what rates are achievable when a sender and a receiver consume a noisy dynamic resource. They additionally consume or generate noiseless classical communication, noiseless quantum communication, and noiseless entanglement. The theorem determines the 3-D "direct-dynamic" capacity region, giving the full tradeoff between the three fundamental noiseless resources when a noisy dynamic resource is available.

A. Proof of the Direct Coding Theorem

The direct coding theorem is the statement that the direct-dynamic capacity region $\mathcal{C}_{\mathrm{DD}}(\mathcal{N})$ contains the direct-dynamic achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DD}}(\mathcal{N})$

$$\tilde{\mathcal{C}}_{\mathrm{DD}}(\mathcal{N}) \subseteq \mathcal{C}_{\mathrm{DD}}(\mathcal{N}).$$

It follows immediately from combining the classically enhanced father resource inequality in (12) with the unit resource inequalities and considering the definition of $\tilde{\mathcal{C}}_{\mathrm{DD}}(\mathcal{N})$ in Theorem 3 and the definition of $\mathcal{C}_{\mathrm{DD}}(\mathcal{N})$ in (11).

B. Proof of the Converse Theorem

The statement of the converse theorem is that the direct-dynamic achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DD}}(\mathcal{N})$ contains the direct-dynamic capacity region

$$\mathcal{C}_{\mathrm{DD}}(\mathcal{N}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}(\mathcal{N}).$$

We consider one octant of the (C, Q, E) space at a time in order to prove the converse theorem.

The converse theorem exploits the converse proofs corresponding to several other protocols: the father protocol [23], [25], classically enhanced quantum communication and entanglement generation [21], classically assisted quantum communication and entanglement generation [34], and entanglement-assisted classical communication [17], [18], [24]. It also exploits some information-theoretic arguments. We briefly review each of these protocols and their relation to the classically enhanced father achievable rate region $\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N})$ in what follows. Fig. 3 illustrates the relation between these protocols.

Entanglement-assisted quantum communication consumes a noisy dynamic resource and noiseless entanglement to generate noiseless quantum communication. The father protocol is a particular protocol for this task. The father's achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DD}}^{0+-}(\mathcal{N})$ and the capacity region $\mathcal{C}_{\mathrm{DD}}^{0+-}(\mathcal{N})$ lie in the Q^{0+-} quadrant of the (C, Q, E) space

$$\tilde{\mathcal{C}}_{\mathrm{DD}}^{0+-}(\mathcal{N}) = \tilde{\mathcal{C}}_{\mathrm{DD}}(\mathcal{N}) \cap Q^{0+-} \quad (52)$$

$$\mathcal{C}_{\mathrm{DD}}^{0+-}(\mathcal{N}) = \mathcal{C}_{\mathrm{DD}}(\mathcal{N}) \cap Q^{0+-}. \quad (53)$$

The above relations shows that these respective regions are special cases of the direct-dynamic achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DD}}(\mathcal{N})$ and the direct-dynamic capacity region $\mathcal{C}_{\mathrm{DD}}(\mathcal{N})$. The capacity theorem states that the capacity region $\mathcal{C}_{\mathrm{DD}}^{0+-}(\mathcal{N})$ is equivalent to the achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DD}}^{0+-}(\mathcal{N})$[25]

$$\mathcal{C}_{\mathrm{DD}}^{0+-}(\mathcal{N}) = \tilde{\mathcal{C}}_{\mathrm{DD}}^{0+-}(\mathcal{N}). \quad (54)$$

The father achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{0+-}(\mathcal{N})$ is a special case of the classically enhanced father protocol where there is no classical communication [31]

$$\tilde{\mathcal{C}}_{\text{DD}}^{0+-}(\mathcal{N}) = \tilde{\mathcal{C}}_{\text{CEF}}(\mathcal{N}) \cap Q^{0+-}. \tag{55}$$

Classically enhanced quantum communication consumes a noisy dynamic resource to generate noiseless classical communication and noiseless quantum communication. The Devetak–Shor protocol is a particular protocol for this task [21]. The achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{++0}(\mathcal{N})$ and the capacity region $\mathcal{C}_{\text{DD}}^{++0}(\mathcal{N})$ lie in the Q^{++0} quadrant of the (C, Q, E) space

$$\tilde{\mathcal{C}}_{\text{DD}}^{++0}(\mathcal{N}) = \tilde{\mathcal{C}}_{\text{DD}}(\mathcal{N}) \cap Q^{++0} \tag{56}$$
$$\mathcal{C}_{\text{DD}}^{++0}(\mathcal{N}) = \mathcal{C}_{\text{DD}}(\mathcal{N}) \cap Q^{++0}. \tag{57}$$

The above relations establish that these respective regions are special cases of the direct-dynamic achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}(\mathcal{N})$ and the direct-dynamic capacity region $\mathcal{C}_{\text{DD}}(\mathcal{N})$. The classically enhanced quantum communication capacity theorem states that the achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{++0}(\mathcal{N})$ is equivalent to the capacity region $\mathcal{C}_{\text{DD}}^{++0}(\mathcal{N})$[21]

$$\tilde{\mathcal{C}}_{\text{DD}}^{++0}(\mathcal{N}) = \mathcal{C}_{\text{DD}}^{++0}(\mathcal{N}). \tag{58}$$

The achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{++0}(\mathcal{N})$ is obtainable from the classically enhanced father's achievable rate region $\tilde{\mathcal{C}}_{\text{CEF}}(\mathcal{N})$ by combining it with ED and keeping the points with zero entanglement [31]

$$\tilde{\mathcal{C}}_{\text{DD}}^{++0}(\mathcal{N}) = (\tilde{\mathcal{C}}_{\text{CEF}}(\mathcal{N}) + L_{\text{ED}}) \cap Q^{++0}. \tag{59}$$

Classically enhanced entanglement generation consumes a noisy dynamic resource to generate noiseless classical communication and noiseless entanglement. The Devetak–Shor protocol is a particular protocol for this task [21]. The achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N})$ and the capacity region $\mathcal{C}_{\text{DD}}^{+0+}(\mathcal{N})$ lie in the Q^{+0+} quadrant of the (C, Q, E) space

$$\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N}) = \tilde{\mathcal{C}}_{\text{DD}}(\mathcal{N}) \cap Q^{+0+} \tag{60}$$
$$\mathcal{C}_{\text{DD}}^{+0+}(\mathcal{N}) = \mathcal{C}_{\text{DD}}(\mathcal{N}) \cap Q^{+0+}. \tag{61}$$

The above relations establish that these respective regions are special cases of the direct-dynamic achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}(\mathcal{N})$ and the direct-dynamic capacity region $\mathcal{C}_{\text{DD}}(\mathcal{N})$. The capacity theorem states that the achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N})$ is equivalent to the capacity region $\mathcal{C}_{\text{DD}}^{+0+}(\mathcal{N})$[21]

$$\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N}) = \mathcal{C}_{\text{DD}}^{+0+}(\mathcal{N}). \tag{62}$$

The achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N})$ is obtainable from the classically enhanced father's achievable rate region $\tilde{\mathcal{C}}_{\text{CEF}}(\mathcal{N})$ by combining it with ED and keeping the points with zero entanglement [31]

$$\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N}) = (\tilde{\mathcal{C}}_{\text{CEF}}(\mathcal{N}) + L_{\text{ED}}) \cap Q^{+0+}. \tag{63}$$

Entanglement-assisted classical communication consumes a noisy dynamic resource and noiseless entanglement to generate noiseless classical communication. The BSST01 protocol is a particular protocol for this task [17]. The achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{+0-}(\mathcal{N})$ and the capacity region $\mathcal{C}_{\text{DD}}^{+0-}(\mathcal{N})$ lie in the Q^{+0-} quadrant of the (C, Q, E) space

$$\tilde{\mathcal{C}}_{\text{DD}}^{+0-}(\mathcal{N}) = \tilde{\mathcal{C}}_{\text{DD}}(\mathcal{N}) \cap Q^{+0-} \tag{64}$$
$$\mathcal{C}_{\text{DD}}^{+0-}(\mathcal{N}) = \mathcal{C}_{\text{DD}}(\mathcal{N}) \cap Q^{+0-}. \tag{65}$$

The above relations establish that these respective regions are special cases of the direct-dynamic achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}(\mathcal{N})$ and the direct-dynamic capacity region $\mathcal{C}_{\text{DD}}(\mathcal{N})$. The capacity theorem states that the achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{+0-}(\mathcal{N})$ is equivalent to the capacity region $\mathcal{C}_{\text{DD}}^{+0-}(\mathcal{N})$[17], [18], [24], [25]

$$\tilde{\mathcal{C}}_{\text{DD}}^{+0-}(\mathcal{N}) = \mathcal{C}_{\text{DD}}^{+0-}(\mathcal{N}). \tag{66}$$

The achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{+0-}(\mathcal{N})$ is obtainable from the classically enhanced father's achievable rate region $\tilde{\mathcal{C}}_{\text{CEF}}(\mathcal{N})$ by combining it with the SD protocol and keeping the points with zero quantum communication [31]

$$\tilde{\mathcal{C}}_{\text{DD}}^{+0-}(\mathcal{N}) = (\tilde{\mathcal{C}}_{\text{CEF}}(\mathcal{N}) + L_{\text{SD}}) \cap Q^{+0-}. \tag{67}$$

Forward classical communication does not increase the entanglement generation capacity or the quantum communication capacity [16], [34]. Thus, there is a simple relation between the achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{-0+}(\mathcal{N})$ and the achievable entanglement generation capacity region $\tilde{\mathcal{C}}_{\text{DD}}^{00+}(\mathcal{N})$

$$\tilde{\mathcal{C}}_{\text{DD}}^{-0+}(\mathcal{N}) = \tilde{\mathcal{C}}_{\text{DD}}^{00+}(\mathcal{N}) + L^{-00}$$

where $L^{-00} \equiv \{\lambda(-1, 0, 0) : \lambda \geq 0\}$. The converse proof of the entanglement generation capacity region states that the achievable rate region $\tilde{\mathcal{C}}_{\text{DD}}^{00+}(\mathcal{N})$ is optimal [8]

$$\tilde{\mathcal{C}}_{\text{DD}}^{00+}(\mathcal{N}) = \mathcal{C}_{\text{DD}}^{00+}(\mathcal{N}).$$

The converse proof of the region $\tilde{\mathcal{C}}_{\text{DD}}^{-0+}(\mathcal{N})$ then follows:

$$\begin{aligned}\mathcal{C}_{\text{DD}}^{-0+}(\mathcal{N}) &= \mathcal{C}_{\text{DD}}^{00+}(\mathcal{N}) + L^{-00} \\ &= \tilde{\mathcal{C}}_{\text{DD}}^{00+}(\mathcal{N}) + L^{-00} = \tilde{\mathcal{C}}_{\text{DD}}^{-0+}(\mathcal{N}).\end{aligned} \tag{68}$$

The entanglement generation achievable rate region results from combining the father achievable rate region in (55) with ED [23]

$$\tilde{\mathcal{C}}_{\text{DD}}^{00+}(\mathcal{N}) = ((\tilde{\mathcal{C}}_{\text{CEF}}(\mathcal{N}) \cap Q^{0+-}) + L_{\text{ED}}) \cap L^{00+}. \tag{69}$$

Similar results for the classically assisted quantum communication capacity region $\mathcal{C}_{\text{DD}}^{-+0}(\mathcal{N})$ hold

$$\begin{aligned}\mathcal{C}_{\text{DD}}^{-+0}(\mathcal{N}) &= \mathcal{C}_{\text{DD}}^{0+0}(\mathcal{N}) + L^{-00} \\ &= \tilde{\mathcal{C}}_{\text{DD}}^{0+0}(\mathcal{N}) + L^{-00} = \tilde{\mathcal{C}}_{\text{DD}}^{-+0}(\mathcal{N}).\end{aligned} \tag{70}$$

The quantum communication achievable rate region results from combining the father achievable rate region in (55) with ED [23]

$$\tilde{\mathcal{C}}_{\mathrm{DD}}^{0+0}(\mathcal{N}) = ((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) \cap Q^{0+-}) + L_{\mathrm{ED}}) \cap L^{0+0}. \quad (71)$$

1) Octants That Generate Quantum Communication: We first consider all of the octants with corresponding protocols that generate quantum communication, i.e., octants of the form $(\pm, +, \pm)$. The proof of one of the octants is the converse proof in [31]. The proofs of two of the remaining three octants are similar to the entanglement-generating octants from the static case. The similarity holds because these octants generate the noiseless version of the noisy dynamic resource. The proof of the last remaining octant is different from any we have seen so far. We discuss later how its proof gives insight into the question of using entanglement-assisted coding versus TP.

$(+, +, -)$. The converse theorem for this octant is the converse for entanglement-assisted communication of classical and quantum information. ([31] contains the proof). We briefly recall these inequalities here

$$\begin{aligned} nC + 2nQ &\leq I(AX; B^n)_\sigma \\ nQ &\leq I(A \rangle B^n X)_\sigma + n|E| \\ nC + nQ &\leq I(X; B^n)_\sigma + I(A \rangle B^n X) + n|E| \end{aligned}$$

where σ is a state of the form in (13). The above inequalities form the one-shot, one-state region. Interestingly, the above set of inequalities represents a translation of the unit resource capacity region to the classically enhanced father protocol.

$(+, +, +)$. This octant exploits the projection technique with ED. Let

$$\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) \equiv \mathcal{C}_{\mathrm{DD}}(\mathcal{N}) \cap O^{+++}$$

and recall the definition of $\mathcal{C}_{\mathrm{DD}}^{+0+}(\mathcal{N})$ in (61). We exploit the line of entanglement distribution L_{ED} as defined in (17). Define the following maps:

$$\begin{aligned} f &: S \rightarrow (S + L_{\mathrm{ED}}) \cap Q^{+0+} \\ \hat{f} &: S \rightarrow (S - L_{\mathrm{ED}}) \cap O^{+++}. \end{aligned}$$

The map f translates the set S in the ED direction and keeps the points that lie on the Q^{+0+} quadrant. The map $\hat{f}$, in a sense, undoes the effect of f by moving the set S back to the O^{+++} octant.

The inclusion $\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) \subseteq \hat{f}(f(\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N})))$ holds because

$$\begin{aligned} \mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) &= \mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) \cap O^{+++} \\ &\subseteq (((\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{+0+}) - L_{\mathrm{ED}}) \cap O^{+++} \\ &= (f(\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N})) - L_{\mathrm{ED}}) \cap O^{+++} \\ &= \hat{f}(f(\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}))). \end{aligned} \quad (72)$$

The first set equivalence is obvious from the definition of $\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N})$. The first inclusion follows from the following logic. Pick any point $a \equiv (C, Q, E) \in \mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N})$ and a particular point $b \equiv (0, -Q, Q) \in L_{\mathrm{ED}}$. It follows that $a + b = (C, 0, E + Q) \in (\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{+0+}$. Pick a point $-b = (0, Q, -Q) \in -L_{\mathrm{ED}}$. It follows that $a + b - b \in (((\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{+0+}) - L_{\mathrm{ED}}) \cap O^{+++}$ and that $a + b - b = (C, Q, E) = a$. The first inclusion thus holds because every point in $\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) \cap O^{+++}$ is in $(((\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{+0+}) - L_{\mathrm{ED}}) \cap O^{+++}$. The second set equivalence follows from the definition of f and the third set equivalence follows from the definition of $\hat{f}$.

It is operationally clear that the following inclusion holds:

$$f(\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N})) \subseteq \mathcal{C}_{\mathrm{DD}}^{+0+}(\mathcal{N}) \quad (73)$$

because the mapping f converts any achievable point $a \in \mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N})$ to an achievable point in $\mathcal{C}_{\mathrm{DD}}^{+0+}(\mathcal{N})$ by consuming all of the quantum communication at point a with ED.

The converse proof of the classically enhanced entanglement generation protocol [21] states that the following inclusion holds:

$$\mathcal{C}_{\mathrm{DD}}^{+0+}(\mathcal{N}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{+0+}(\mathcal{N}). \quad (74)$$

The inclusion $\hat{f}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{+0+}(\mathcal{N})) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{+++}(\mathcal{N})$ holds because

$$\begin{aligned} &\hat{f}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{+0+}(\mathcal{N})) \\ &= (((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{+0+}) - L_{\mathrm{ED}}) \cap O^{+++} \\ &\subseteq ((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{ED}}) - L_{\mathrm{ED}}) \cap O^{+++} \\ &= ((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{ED}}) \cap O^{+++}) \\ &\quad \cup ((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) - L_{\mathrm{ED}}) \cap O^{+++}) \\ &\subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{+++}(\mathcal{N}). \end{aligned} \quad (75)$$

The first set equivalence follows from (63) the definition of $\hat{f}$. The first inclusion follows by dropping the intersection with Q^{+0+}. The second set equivalence follows because $(\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{ED}}) - L_{\mathrm{ED}} = (\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{ED}}) \cup (\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) - L_{\mathrm{ED}})$, and the second inclusion holds by (14) and because $(\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) - L_{\mathrm{ED}}) \cap O^{+++} = (0, 0, 0)$.

Putting (72)–(75) together, we have the following inclusion:

$$\begin{aligned} \mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) &\subseteq \hat{f}(f(\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}))) \\ &\subseteq \hat{f}(\mathcal{C}_{\mathrm{DD}}^{+0+}(\mathcal{N})) \subseteq \hat{f}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{+0+}(\mathcal{N})) \\ &\subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{+++}(\mathcal{N}). \end{aligned}$$

The above inclusion $\mathcal{C}_{\mathrm{DD}}^{+++}(\mathcal{N}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{+++}(\mathcal{N})$ is the statement of the converse theorem for this octant.

The proof of the octant $(-, +, +)$ is similar to the above proof and appears in Appendix IV.

$(-, +, -)$. This octant is one of the more interesting octants for the direct-dynamic tradeoff. Interestingly, this octant gives insight into the question of the use of TP versus entanglement-assisted quantum error correction [37]–[45]. We discuss these insights in Section VII-C. The converse proof for this octant follows from a *reductio ad absurdum* argument, reminiscent of similar arguments that appeared in [31]. We use this technique to obtain two bounds for this octant, obviating the need to consider the technique used for the previous octants. Fig. 4 illustrates the bounds for this octant.

First, consider that the bound $nQ \leq I(A \rangle B^n X)$ applies to all points in the Q^{-+0} quadrant because forward classical

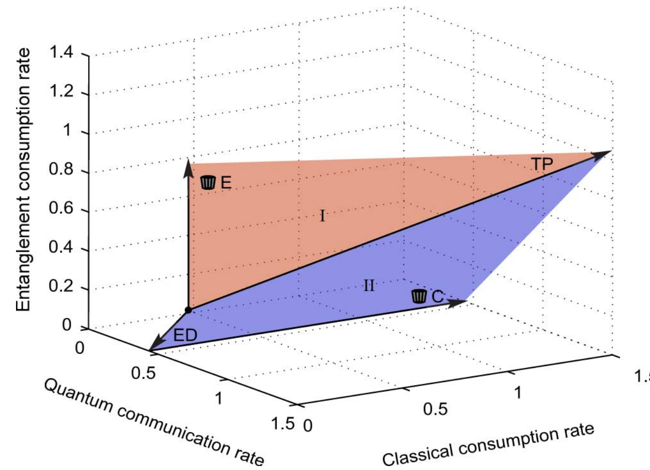


Fig. 4. Boundary of the region in the octant $(-,+,-)$ that corresponds to classical- and entanglement-assisted quantum communication. Plane I corresponds to the bound in (77), and plane II corresponds to the bound in (76). There cannot be any points outside plane II. If there were, we could combine such points with ED to beat the bound on the quantum communication capacity. There cannot be any points inside plane II but outside plane I. If there were, we could combine such points with SD to beat the entanglement-assisted quantum capacity bound. Planes I and II intersect at the line of TP. We can achieve all points inside bounds (76) and (77) simply by combining the father protocol (the black dot) with TP and ED.

communication does not increase the quantum communication capacity of a quantum channel [16], [34]. Consider combining achievable points along the boundary of the above bound with the inverse of the ED protocol. This procedure outlines the following bound:

$$nQ \leq I\left(A\rangle B^{n}X\right) + n\left|E\right|. \tag{76}$$

The bound in (76) applies to all points in this octant. Were it not so, then one could combine points outside (76) with ED and beat the bound $nQ \leq I\left(A\rangle B^{n}X\right)$ that applies to all points in the Q^{-+0} quadrant.

Next, consider that the bound $2nQ \leq I\left(AX;B^{n}\right)$ applies to all points in the Q^{0+-} quadrant. This bound follows from the limit on the entanglement-assisted quantum capacity of a quantum channel [17], [25], [31]. Consider combining achievable points along the boundary of the above bound with the inverse of the SD protocol. This procedure outlines the following bound:

$$2nQ \leq I\left(AX;B^{n}\right) + n\left|C\right|. \tag{77}$$

The bound in (77) applies to all points in this octant. Were it not so, then one could combine points inside (76) but outside (77) with SD and beat the bound $2nQ \leq I\left(AX;B^{n}\right)$ that applies to all points in the Q^{0+-} quadrant.

We can achieve all points specified by the above boundaries simply by combining the classically enhanced father protocol with TP or ED. Thus, the statement of the converse holds for this octant

$$\mathcal{C}_{\text{DD}}^{-+-}\left(\mathcal{N}\right) \subseteq \tilde{\mathcal{C}}_{\text{DD}}^{-+-}\left(\mathcal{N}\right).$$

Fig. 4 illustrates these arguments.

We can also obtain these bounds with a direct, information-theoretic argument. This argument appears in Appendix VI.

Remark 1: The two bounds in (76) and (77) are the same as two bounds that apply to the octant O^{++-} (corresponding to entanglement-assisted communication of classical and quantum information [31]). Thus, these bounds are extensions of the same bounds from the O^{++-} octant.

2) Octants That Consume Quantum Communication: We now consider the four octants with corresponding protocols that consume quantum communication, i.e., octants of the form $(\pm,-,\pm)$. The proof of one of the octants is trivial. The proofs of two of the remaining three octants are similar to the proofs of the entanglement-consuming octants from the static case. The similarity holds because these octants consume the noiseless version of the noisy dynamic resource. The proof of the other octant exploits information theoretic arguments.

$(-,-,-)$. $\tilde{\mathcal{C}}_{\text{DD}}\left(\mathcal{N}\right)$ completely contains this octant.

$(+,-,+)$. We use a trick similar to the $(+,-,-)$ octant for the static case. Let $\text{id}^{\otimes|Q|}$ denote a noiseless qubit channel of size $|Q|$ qubits. We need the following additivity lemma.

Lemma 4: The following inclusion holds:

$$\mathcal{C}_{\text{DD}}^{+0+}(\mathcal{N} \otimes \text{id}^{\otimes|Q|}) \subseteq \tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\text{id}^{\otimes|Q|}).$$

Proof: ED induces a linear bijection $f : \tilde{\mathcal{C}}_{\text{DD}}^{++0}\left(\mathcal{N}\right) \rightarrow \tilde{\mathcal{C}}_{\text{DD}}^{+0+}\left(\mathcal{N}\right)$ between the classically enhanced quantum communication achievable region $\tilde{\mathcal{C}}_{\text{DD}}^{++0}$ and the classically enhanced entanglement generation achievable region $\tilde{\mathcal{C}}_{\text{DD}}^{+0+}$. The linear bijection f behaves as follows for every point $(R,Q,0) \in \tilde{\mathcal{C}}_{\text{DD}}^{++0}$:

$$f : (R,Q,0) \rightarrow (R,0,Q).$$

The following relation holds:

$$f(\tilde{\mathcal{C}}_{\text{DD}}^{++0}\left(\mathcal{N}\right)) = \tilde{\mathcal{C}}_{\text{DD}}^{+0+}\left(\mathcal{N}\right) \tag{78}$$

because applying ED to the classically enhanced quantum communication resource inequality gives classically enhanced entanglement generation [21]. The inclusion $\mathcal{C}_{\text{DD}}^{++0}(\mathcal{N}\otimes\text{id}^{\otimes|Q|}) \subseteq f^{-1}(\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\text{id}^{\otimes|Q|}))$ holds because

$$\begin{aligned}
\mathcal{C}_{\text{DD}}^{++0}(\mathcal{N} \otimes \text{id}^{\otimes|Q|}) &= \mathcal{C}_{\text{DD}}^{++0}(\mathcal{N}) + (0,Q,0) \\
&\subseteq \mathcal{C}_{\text{DD}}^{++0}(\mathcal{N}) + \mathcal{C}_{\text{DD}}^{++0}(\text{id}^{\otimes|Q|}) \\
&= \tilde{\mathcal{C}}_{\text{DD}}^{++0}(\mathcal{N}) + \tilde{\mathcal{C}}_{\text{DD}}^{++0}(\text{id}^{\otimes|Q|}) \\
&= f^{-1}(\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N})) \\
&\quad + f^{-1}(\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\text{id}^{\otimes|Q|})) \\
&= f^{-1}(\tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\text{DD}}^{+0+}(\text{id}^{\otimes|Q|})).
\end{aligned}$$

The first set equivalence follows because the capacity region of the noisy channel $\mathcal{N}$ combined with a rate Q noiseless qubit channel is equivalent to a translation of the capacity region of the noisy channel $\mathcal{N}$. The first inclusion follows because the capacity region of a rate Q noiseless qubit channel contains the rate triple $(0,Q,0)$. The second set equivalence follows from the classically enhanced quantum communication capacity theorem in (58), the third set equivalence from (78), and the fourth set equivalence from linearity of the map f. The above inclusion implies the following one:

$$f(\mathcal{C}_{\rm DD}^{++0}(\mathcal{N}\otimes \mathrm{id}^{\otimes|Q|}))\subseteq \tilde{\mathcal{C}}_{\rm DD}^{+0+}(\mathcal{N})+\tilde{\mathcal{C}}_{\rm DD}^{+0+}(\mathrm{id}^{\otimes|Q|}).$$

The lemma follows because

$$\begin{aligned} f(\mathcal{C}_{\rm DD}^{++0}(\mathcal{N}\otimes \mathrm{id}^{\otimes|Q|})) &= f(\tilde{\mathcal{C}}_{\rm DD}^{++0}(\mathcal{N}\otimes \mathrm{id}^{\otimes|Q|}))\\ &=\tilde{\mathcal{C}}_{\rm DD}^{+0+}(\mathcal{N}\otimes \mathrm{id}^{\otimes|Q|})\\ &=\mathcal{C}_{\rm DD}^{+0+}(\mathcal{N}\otimes \mathrm{id}^{\otimes|Q|})\end{aligned}$$

where we apply the relations in (58), (78), and (62). □

We now begin the converse proof of this octant. Observe that

$$\tilde{\mathcal{C}}_{\rm DD}^{+0+}(\mathrm{id}^{\otimes|Q|})=\tilde{\mathcal{C}}_{\rm U}^{+Q+}. \tag{79}$$

Hence, for all $Q\leq 0$

$$\begin{aligned}\mathcal{C}_{\rm DD}^{+Q+}(\mathcal{N}) &=\mathcal{C}_{\rm DD}^{+0+}(\mathcal{N}\otimes \mathrm{id}^{\otimes|Q|})\\ &\subseteq\tilde{\mathcal{C}}_{\rm DD}^{+0+}(\mathcal{N})+\tilde{\mathcal{C}}_{\rm U}^{+Q+}\end{aligned} \tag{80}$$

where we apply Lemma 4 and (79). The inclusion $\mathcal{C}_{\rm DD}^{+-+}(\mathcal{N})\subseteq\tilde{\mathcal{C}}_{\rm DD}^{+-+}$ follows because

$$\begin{aligned}\mathcal{C}_{\rm DD}^{+-+}(\mathcal{N}) &=\bigcup_{Q\leq 0}\mathcal{C}_{\rm DD}^{+Q+}(\mathcal{N})\\ &\subseteq\bigcup_{Q\leq 0}(\tilde{\mathcal{C}}_{\rm DD}^{+0+}(\mathcal{N})+\tilde{\mathcal{C}}_{\rm U}^{+Q+})\\ &=(\tilde{\mathcal{C}}_{\rm DD}^{+0+}(\mathcal{N})+\tilde{\mathcal{C}}_{\rm U})\cap O^{+-+}\\ &\subseteq(\tilde{\mathcal{C}}_{\rm CEF}(\mathcal{N})+\tilde{\mathcal{C}}_{\rm U})\cap O^{+-+}\\ &=\tilde{\mathcal{C}}_{\rm DD}^{+-+}(\mathcal{N}).\end{aligned}$$

The first and equivalence hold by definition, the first inclusion follows from (80), the second set equivalence follows because $\bigcup_{Q\leq 0}\tilde{\mathcal{C}}_{\rm U}^{+Q+}=\tilde{\mathcal{C}}_{\rm U}\cap O^{+-+}$, and the second inclusion follows because combining the classically enhanced father region with ED gives the region for classically enhanced entanglement generation. The above inclusion $\mathcal{C}_{\rm DD}^{+-+}(\mathcal{N})\subseteq\tilde{\mathcal{C}}_{\rm DD}^{+-+}$ is the statement of the converse theorem for this octant.

The proof of the $(-,-,+)$ octant is similar to the proof of the above octant and appears in Appendix V.

$(+,-,-)$. The converse proof for this octant exploits information-theoretic arguments to show that the following bounds apply to all rate triples $(C,-|Q|,-|E|)$ in this octant:

$$nC\leq I(AX;B^n)+2n|Q| \tag{81}$$
$$nC\leq I(X;B^n)+I(A\rangle B^nX)+n|Q|+n|E|. \tag{82}$$

Fig. 5 depicts the most general protocol that generates classical communication with the help of a noisy channel, noiseless entanglement, and noiseless quantum communication. The state at the beginning of the protocol is as follows:

$$\omega^{MM'T_AT_B}\equiv\frac{1}{M}\sum_m|m\rangle\langle m|^M\otimes|m\rangle\langle m|^{M'}\otimes\Phi^{T_AT_B}.$$

Alice performs an encoding according to some CPTP map $\mathcal{E}^{M'T_A\to A'^nA_1}$ that takes the classical system M' of size 2^{nC} and the quantum system T_A as input. The map $\mathcal{E}^{M'T_A\to A'^nA_1}$ outputs some systems A'^n and a quantum state in a register A_1.

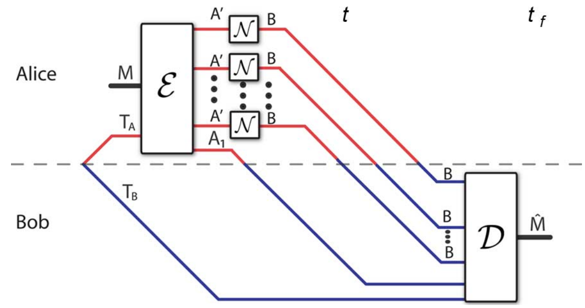


Fig. 5. The most general protocol for classical communication with the help of a noisy channel, noiseless entanglement, and noiseless quantum communication. Alice wishes to communicate a classical message M to Bob. She shares entanglement with Bob in the form of maximally entangled states. Her half of the entanglement is in system T_A and Bob's half is in the system T_B. Alice performs some CPTP encoding map $\mathcal{E}$ on her classical message and her half of the entanglement. The output of this encoder is a quantum state in some register A_1 and a large number of systems A'^n that are input to the channel. She transmits A'^n through the noisy channel and the system A_1 over noiseless qubit channels. Bob receives the outputs B^n of the channel and the system A_1 from the noiseless qubit channels. He combines these with his half of the entanglement and decodes the classical message that Alice transmits.

The Schmidt rank of the shared entanglement is 2^{nE} and the size of the quantum system is 2^{nQ}. The state after the encoder is as follows:

$$\omega^{MA'^nA_1T_B}\equiv\mathcal{E}^{M'T_A\to AA_1}(\omega^{MM'T_AT_B}).$$

She then transmits the above state through many uses of the noisy channel $\mathcal{N}$, producing the following state at time t:

$$\omega^{MB^nA_1T_B}\equiv\mathcal{N}^{A'^n\to B^n}(\omega^{MA'^nA_1T_B}). \tag{83}$$

Note that the state at this point is near to being a state of the form in (13), if we define $A\equiv A_1T_B$ (more on this later). Finally, Bob combines all of his systems and passes them through a CPTP decoding map $\mathcal{D}^{B^nA_1T_B\to\hat{M}}$ that produces his estimate $\hat{M}$ of the classical message M. A protocol is ϵ-good if the probability of decoding the classical message incorrectly is low

$$\Pr\left\{\hat{M}\neq M\right\}\leq\epsilon. \tag{84}$$

We first prove the bound in (81). Consider the following chain of inequalities:

$$\begin{aligned} nC &= H(M)\\ &=I(M;\hat{M})+H(M|\hat{M})\\ &\leq I(M;\hat{M})+n\delta'\\ &\leq I(M;A_1B^nT_B)_\omega+n\delta'\\ &=I(M;B^n|A_1T_B)_\omega+I(M;A_1T_B)_\omega+n\delta'\\ &=I(A_1T_BM;B^n)_\omega-I(A_1T_B;B^n)_\omega\\ &\quad+I(M;A_1T_B)_\omega+n\delta'\\ &\leq I(A_1T_BM;B^n)_\omega+I(M;A_1T_B)_\omega+n\delta'\\ &=I(A_1T_BM;B^n)_\omega+I(M;T_B)_\omega\\ &\quad+I(A_1;T_B|M)_\omega+n\delta'\\ &=I(A_1T_BM;B^n)_\omega+H(A_1|M)_\omega\\ &\quad-H(A_1|T_BM)_\omega+n\delta'\\ &\leq I(AM;B^n)_\omega+n2|Q|+n\delta'.\end{aligned}$$

The first equality follows because the classical message M is uniform, and the second equality follows by a straightforward entropic manipulation. The first inequality follows from the condition in (84) and Fano's inequality [51] with $\delta' \equiv \epsilon C + H_2(\epsilon)/n$. The second inequality follows from the quantum data processing inequality [35]. The third and fourth equalities follow by expanding the quantum mutual information with the chain rule. The third inequality follows because $I(A_1 T_B; B^n)_\omega \geq 0$. The fifth equality follows by expanding the mutual information $I(M; A_1 T_B)_\omega$ with the chain rule. The last equality follows because $I(M; T_B)_\omega = 0$ for this protocol and $I(A_1; T_B|M)_\omega = H(A_1|M)_\omega - H(A_1|T_B M)_\omega$. The final inequality follows from the definition $A \equiv A_1 T_B$ and because $H(A_1|M)_\omega \leq nQ$ and $|H(A_1|T_B M)_\omega| \leq nQ$.

We now prove the bound in (82). Consider the following chain of inequalities:

$$\begin{aligned} nC \leq\, & I(A_1 T_B M; B^n)_\omega + I(A_1; T_B|M)_\omega + n\delta' \\ =\, & I(M; B^n)_\omega + I(A_1 T_B; B^n|M)_\omega + H(A_1|M)_\omega \\ & + H(T_B|M)_\omega - H(A_1 T_B|M)_\omega + n\delta' \\ =\, & I(M; B^n)_\omega + I(A_1 T_B \rangle B^n M)_\omega \\ & + H(A_1|M)_\omega + H(T_B|M)_\omega + n\delta' \\ \leq\, & I(M; B^n)_\omega + I(A\rangle B^n M)_\omega + n|Q| + n|E| + n\delta'. \end{aligned}$$

The first inequality follows from the fifth equality above and the fact that $I(M; T_B)_\omega = 0$ for this protocol. The first equality follows by applying the chain rule for quantum mutual information to $I(A_1 T_B M; B^n)_\omega$ and by expanding the mutual information $I(A_1; T_B|M)_\omega$. The second equality follows by noting that

$$I(A_1 T_B; B^n|M)_\omega = I(A_1 T_B \rangle B^n M)_\omega + H(A_1 T_B|M)_\omega .$$

The last inequality follows from the definition $A \equiv A_1 T_B$ and the fact that the entropies $H(A_1|M)_\omega$ and $H(T_B|M)_\omega$ are less than the logarithm of the dimensions of the respective systems A_1 and T_B.

We should make some final statements concerning this proof. The state in (83) as we have defined it is not quite a state of the form in (13) because the encoder could be a general CPTP map. Though, a few arguments demonstrate that a collection of isometric maps works just as well as a general CPTP map, and it then follows that the state in (83) is of the form in (13). First, consider that a general CPTP map applied to a classical-quantum state of the form in (83) merely acts as a collection of CPTP maps $\{\mathcal{E}_m^{T_A \rightarrow AA_1}\}$ indexed by the classical message m (see [46, Sec. 2.3.7]). Each of these CPTP maps has an isometric extension to some purifying system E'. Alice can then simulate these maps with the isometries $\{U_{\mathcal{E}_m}^{T_A \rightarrow AA_1E'}\}$ but she instead performs a complete von Neumann measurement of the system E', producing a classical system Y. By the same arguments as in [25, Th. 7.8] and [31, App. E], the protocol can only improve under this measurement, so that it is sufficient to consider isometric encoders. Thus, the state in (83) is a state of the form in (13) and this concludes the converse proof for this octant.

C. Discussion

The proof of the octant O^{-+-} is one of the more interesting octants in the direct-dynamic tradeoff. Its proof directly answers the following question concerning the use of entanglement-assisted quantum error correction [37]–[45]:

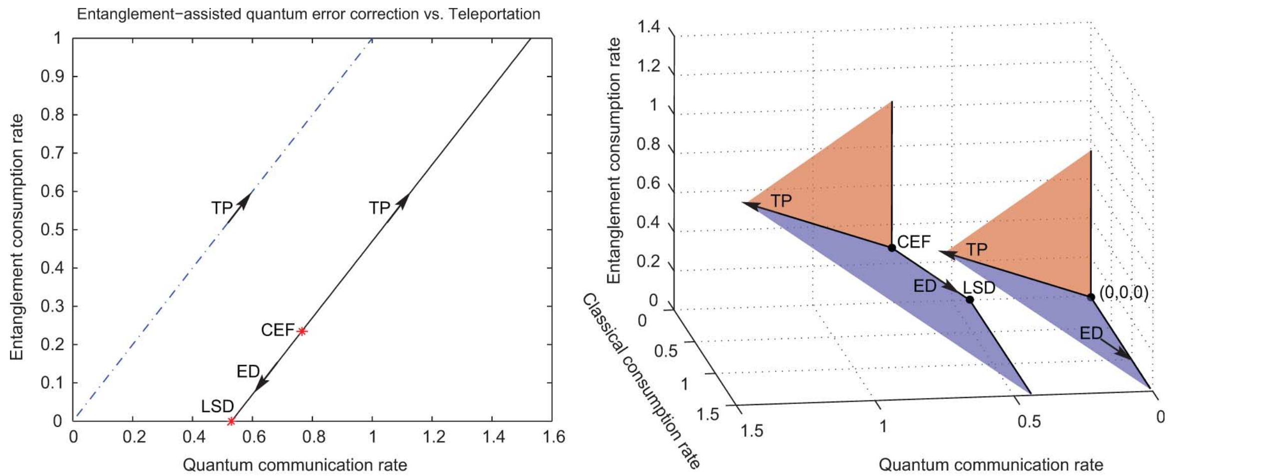
Why even use entanglement-assisted coding if TP is a way to consume entanglement for the purpose of transmitting quantum information?

The proof of the octant O^{-+-} gives a practical answer to the above question by showing exactly how entanglement-assisted quantum error correction is useful. We illustrate our arguments by considering the specific case of a dephasing qubit channel with dephasing parameter $p = 0.2$. The quantum capacity of this channel is around 0.5 qubits per channel use.

First, let us suppose that classical communication is a free resource. Then, we can project the boundary of the octant O^{-+-} and the line of TP into the quadrant Q^{0+-} to compare entanglement-assisted quantum coding to TP. Fig. 6(a) illustrates this projection. From this figure, we observe that the superior strategy is to combine quantum communication (LSD) with TP or to combine the classically enhanced father protocol (CEF) with TP. If we do not take advantage of coding quantum information over the channel, we have to consume more entanglement in order to achieve the same amount of quantum communication. The figure demonstrates that a naive strategy employing TP alone must consume around 0.5 more ebits per channel use for the same amount of quantum communication that one can obtain by combining LSD or CEF with TP—this result holds for the qubit dephasing channel with dephasing parameter 0.2.

Next, let us consider the case when classical communication is not free. Then, we must consider the full achievable rate region in the octant O^{-+-}. Fig. 6(b) depicts this scenario by showing both the achievable rate region that combines the classically enhanced father protocol with TP and ED and the achievable rate region that combines TP and ED only. We observe that the second achievable rate region is strictly inside the first one whenever the channel has a positive entanglement-assisted quantum capacity. Thus, the best strategy is not merely to teleport, but it is to combine channel coding (the classically enhanced father protocol) with the two unit protocols of TP and ED. Furthermore, with channel coding, one consumes less entanglement or classical communication in order to achieve a given rate of quantum communication (this improvement can be dramatic if the entanglement-assisted quantum capacity of the channel is large).

Sometimes, quantum Shannon theory gives insight into practical quantum error correction schemes. Devetak's proof of the quantum channel coding theorem shows that codes with a CSS-like structure are good enough for achieving capacity [8]. Another case occurs with the classically enhanced father protocol [31], regarding the structure of classically enhanced entanglement-assisted quantum codes, and yet another occurs in multiple-access quantum coding [52], regarding the structure of multiple-access quantum codes. This octant proves to be another case where quantum Shannon theory gives some interesting guidelines for the optimal strategy of a quantum error correction scheme.



(a) (b)

Fig. 6. (a) Projection of the octant O^{-+-} into the quadrant Q^{0+-}. The figure shows the projection of the line of TP in this plane and the projection of the line of TP starting at the classically enhanced father achievable point. The plot demonstrates that a naive strategy that only teleports and does not take advantage of channel coding requires a higher rate of entanglement consumption in order to achieve a given rate of quantum communication. The better strategy is to employ channel coding (either quantum coding or entanglement-assisted quantum coding). (b) Full capacity region of a strategy that exploits channel coding and the full capacity region of one that does not take advantage of channel coding. The latter capacity region is always strictly inside the former whenever the channel has a positive entanglement-assisted quantum capacity, demonstrating that it is more useful to take advantage of channel coding.

VIII. Single-Letter Examples

In this section, we give an example of a shared state for which the static region single-letterizes and an example of a noisy channel for which the dynamic region single-letterizes. Single-letterization implies that we do not have to consider many copies or many channel uses to compute the respective capacity regions—we only have to consider one copy or channel use, implying that the computation of the region is a tractable problem.

A. Static Case

We consider an "erased state" as our example. We first show that the $(-,-,+)$ octant single-letterizes. Then, the full static region single-letterizes by our previous arguments above.

Suppose that the state that Alice and Bob share is the following erased version of a maximally entangled Bell state:

$$\rho^{AB} \equiv (1-\epsilon)\left(\Phi^{+}\right)^{AB} + \epsilon\pi^{A} \otimes |e\rangle\langle e|^{B}$$

where

$$\left|\Phi^{+}\right\rangle^{AB} \equiv \frac{1}{\sqrt{2}}\left(|00\rangle^{AB} + |11\rangle^{AB}\right).$$

This state arises from sending Alice's share of the state $\left|\Phi^{+}\right\rangle^{AB}$ through an erasure channel that acts as

$$\sigma \longrightarrow (1-\epsilon)\,\sigma + \epsilon\,|e\rangle\langle e|.$$

In what follows, all entropies are with respect to the state ρ

$$H(A) = 1$$
$$H(B) = 1 - \epsilon + H_2(\epsilon)$$
$$H(AB) = H(E) = \epsilon + H_2(\epsilon).$$

Then, the following information quantities appearing in the mother protocol [25] and entanglement distillation [16] are as follows:

$$I(A\rangle B) = 1 - 2\epsilon$$
$$\frac{1}{2}I(A;B) = 1 - \epsilon$$
$$\frac{1}{2}I(A;E) = \epsilon.$$

Let us first consider the plane of classically assisted entanglement distillation. We can achieve the point $(2\epsilon, 0, 1-2\epsilon)$ by the hashing protocol [16] (the classical communication rate required to achieve this distillation yield is $I(A;E) = 2\epsilon$). The rate of entanglement distillation can be no higher than $1-2\epsilon$, which one can actually prove via the quantum capacity theorem (observe that the maximally entangled state maximizes the coherent information and classical communication does not increase the entanglement generation capacity). Thus, we know that the bound $E \leq 1-2\epsilon$ applies for all $C \geq 2\epsilon$ and $Q = 0$. Now we should prove that timesharing between the origin and the point $(2\epsilon, 0, 1-2\epsilon)$ is an optimal strategy.

Consider a scheme of entanglement distillation for an erased state with erasure parameter ϵ. If each party has n halves of the shared states, then Bob shares $n(1-\epsilon)$ ebits with Alice and the environment shares $n\epsilon$ ebits with Alice (for the case of large n). From these $n(1-\epsilon)$ shared ebits, Alice and Bob can perform local operations and forward classical communication to distill $n(1-2\epsilon)$ logical ebits, by the entanglement distillation result for the erased state. This implies an optimal "decoding ratio" of $n(1-2\epsilon)$ decoded ebits for the $n(1-\epsilon)$ physical ebits: $(1-2\epsilon)/(1-\epsilon)$. Now let us consider some strategy for the erased state that mixes between the forward classical communication rate of 2ϵ and no forward classical communication. Suppose that they can achieve the rate triple $(\lambda 2\epsilon, 0, \lambda(1-2\epsilon)+\delta)$ where δ is some small positive number and $0 \leq \lambda \leq 1$ (so that this rate triple represents any point that beats the timesharing limit). Now if they share n erased states, Alice and Bob share $n(1-\epsilon)$ ebits and the environment again shares $n\epsilon$ of them with Alice. But this time, Alice and Bob are not allowed to perform forward classical communication on $n(1-\lambda)(1-\epsilon)$ of them (or a subspace of them of this size). Thus, these states are not available

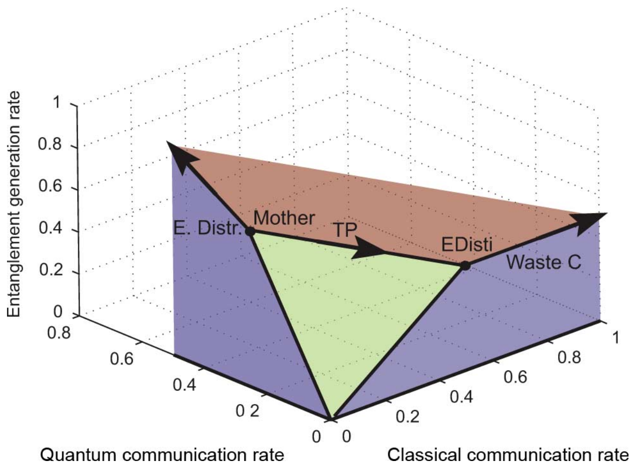


Fig. 7. Plot of the $(-,-,+)$ octant of the static capacity region for the case of an "erased state." The region does not exhibit a tradeoff, and timesharing between the mother protocol (the point labeled "Mother"), entanglement distillation (the point labeled "Edisti"), and the origin is the optimal strategy.

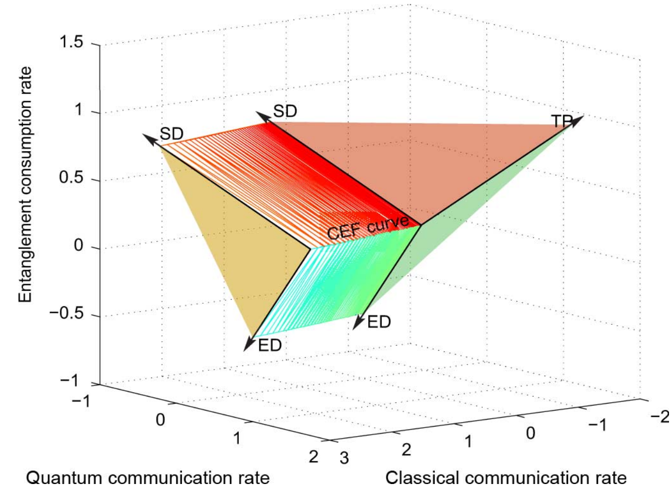


Fig. 8. Plot of the full dynamic capacity region for all octants for the case of a qubit dephasing channel with dephasing parameter $p = 0.2$. We can see that it is merely the unit resource capacity region translated along the classically enhanced father tradeoff curve.

for decoding ebits. This leaves $n(1-\epsilon) - n(1-\lambda)(1-\epsilon) = n\lambda(1-\epsilon)$ qubits available for decoding the ebits. If Alice and Bob could decode $n(\lambda(1-2\epsilon)+\delta)$ logical ebits by local operations and forward classical communication, this would contradict the optimality of the above "decoding ratio" because $n(\lambda(1-2\epsilon)+\delta)/(n\lambda(1-\epsilon)) = (1-2\epsilon)/(1-\epsilon)+\delta/\lambda(1-\epsilon)$ is greater than the optimal decoding ratio $(1-2\epsilon)/(1-\epsilon)$. Therefore, they must only be able to decode $n(1-\lambda)(1-2\epsilon)$ logical ebits. This proves that timesharing between entanglement distillation and the origin is an optimal strategy for the erased state.

The above argument then gives that the following region in the $-0+$ plane is optimal (note that we keep the convention that the rate C is positive even though the protocol consumes it):

$$E \leq 1 - 2\epsilon, \qquad \text{if } C \geq 2\epsilon$$
$$E \leq \frac{C}{2\epsilon}(1-2\epsilon), \qquad \text{if } C < 2\epsilon.$$

We can then obtain a bound on the whole $(-,-,+)$ octant by extending this region by "inverse" TP. That is, the above region, combined with inverse TP, gives a bound on all points in the grandmother octant. Were it not so, then one could combine points outside this bound with TP and achieve points outside the above region, contradicting the optimality of the region.

For achievability, we can achieve all points in the $(-,-,+)$ static octant of the erased state by combining the mother point $(\epsilon, 0, 1-\epsilon)$ with TP $(2Q, -Q, -Q)$ and the wasting of classical communication and quantum communication. Fig. 7 plots this region for the case of an erased state with parameter $\epsilon = 1/4$. It follows that the full region single-letterizes for the case of an erased state, by our characterization of the static region in Theorem 2.

B. Dynamic Case

Our example for the dynamic case is the qubit dephasing channel with dephasing parameter p. We showed in [31] and [32] that the $(+,+,-)$ octant single-letterizes for this channel. The characterization of the dynamic capacity region in Theorem 3 shows that the classically enhanced father protocol combined with the unit protocols achieves the full region. Thus, it is sufficient to determine the classically enhanced father over one use of the channel, and we only need a single use to get the full capacity region. Also, [33] in fact gives a simplified, direct argument that this is the case.

IX. Conclusion

We have provided a unifying treatment of many of the important results in quantum Shannon theory. Our first result is a solution of the unit resource capacity region—the optimal strategy mixes SD, TP, and ED. Our next result is the full triple tradeoff for the static scenario where a sender and a receiver share a noisy bipartite state. The coding strategy combines the classically assisted state redistribution protocol with the three unit protocols. Our last result is a solution of the direct-dynamic capacity theorem—the scenario where a sender and a receiver have access to a large number of independent uses of a noisy quantum channel. The coding strategy combines the classically enhanced father protocol with the three unit protocols.

The discussion in Section VII-C demonstrates another case where quantum Shannon theory has practical implications for quantum error correction schemes. We are able to determine how one benefits from entanglement-assisted coding versus TP.

Our work was originally inspired by the work in [53] in which the authors solved a triple tradeoff problem called generalized remote state preparation (GRSP). The relation between our capacity regions and theirs is yet unknown due to incompatible definitions of a resource [25]. The GRSP uses "pseudoresources" that resemble our definition of a resource but fail to satisfy the quasi-i.i.d. requirement. We can possibly remedy this by generalizing our definition of a resource.

In this paper, we have discussed the triple tradeoff scenario for when a protocol consumes a noisy resource to generate noiseless resources. An interesting open research question is the triple tradeoff scenario for when a protocol generates or *simulates* a noisy resource rather than consumes it. A special case of this type of triple tradeoff is the quantum reverse Shannon theorem, because the protocol corresponding to it consumes classical communication and entanglement to simulate a noisy channel [26], [54], [55]. The discussion in the last section of [17] speaks of the usefulness of the quantum reverse Shannon theorem and its role in simplifying quantum Shannon theory. One could imagine several other protocols that would arise as special cases of the triple tradeoff for simulating a noisy resource, but the usefulness of such triple tradeoffs is unclear to us at this point.

Another interesting open research question concerns the triple tradeoffs for the static and dynamic cases where the noiseless resources are instead public classical communication, private classical communication, and a secret key. We expect the proof strategies to be similar to those in this paper, but the capacity regions should be different from those found here. A useful protocol is the publicly enhanced secret-key-assisted private classical communication protocol [56], an extension of the private father protocol [57]. This protocol gives the initial steps for finding the full triple tradeoff of the dynamic case. The static case should employ previously found protocols, such as that for secret key distillation. As a last suggestion, one might also consider using these techniques for determining the optimal sextuple tradeoffs in multiple-access coding [52], [58] and broadcast channel coding [59], [60].

A last interesting open research question concerning our results is the long-standing "single-letterizable" issue that plagues most capacity results in quantum Shannon theory. Our capacity formulas are regularized expressions—the implication of regularization is that the evaluation of the rate region with a regularized expression is intractable, requiring an optimization over an infinite number of uses of a channel for the dynamic case and over all instruments for the static case. Additionally, prior work shows that two different regularized capacity expressions can coincide asymptotically [52], even though the corresponding finite capacity formulas trace out different finite approximations of the rate region. Thus, regularized results in quantum Shannon theory present a problem for determining the true characterization of a capacity region. We have shown examples of states and channels for which the regions single-letterize, but there is always the possibility of uncovering some formulas for the region that provide a single-letter characterization.

Appendix I
$(-,-,+)$ Octant of the Direct-Static Capacity Region

The converse proof for this octant corresponds to the classically assisted state redistribution protocol. We employ an information-theoretic argument.

We consider the most general classically assisted state redistribution protocol for proving the converse theorem (illustrated in Fig. 9).

Suppose Alice and Bob share many copies $\rho^{A^n B^n}$ of the noisy bipartite state ρ^{AB}. The purification of the state $\rho^{A^n B^n}$ is $\psi^{A^n B^n E^n}$ where E is the reference party. Alice performs a quantum instrument $\tilde{\mathcal{T}}^{A^n \rightarrow A' A_1 M}$ on her system A^n and produces the quantum systems A' and A_1 and the classical system M. The quantum system A' has size 2^{nQ}, A_1 has size 2^{nE}, and M has size 2^{nC}. We consider an extension $\tilde{\mathcal{T}}^{A^n \rightarrow A' A_1 M E'}$ of the quantum instrument $\tilde{\mathcal{T}}^{A^n \rightarrow A' A_1 M}$ in what follows (see the

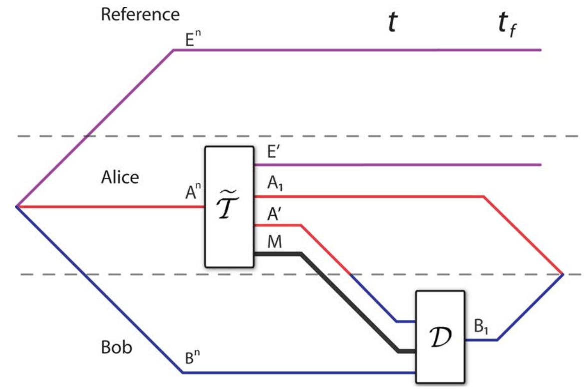


Fig. 9. Most general protocol for classically and quantum-communication-assisted entanglement distillation. Alice, Bob, and the reference system share a state $|\psi\rangle^{A^n B^n E^n}$. Alice performs a quantum instrument $\mathcal{T}^{A^n \rightarrow A' A_1 M E'}$ on her system A^n. Alice transmits M and A_1 to Bob. Bob performs a decoding operation $\mathcal{D}^{B^n M A' \rightarrow B_1}$ that outputs the system B_1. The result of the protocol is a state close to the maximally entangled state $\Phi^{A_1 B_1}$ on systems A_1 and B_1.

discussion of the CP formalism in [25]). Alice sends A' through a noiseless quantum channel and sends M through a noiseless classical channel. The resulting state, at time t in Fig. 9, is as follows:

$$\omega^{A' A_1 M E' B^n E^n} \equiv \tilde{\mathcal{T}}^{A^n \rightarrow A' A_1 M E'} \left(\psi^{A^n B^n E^n} \right).$$

Define $A \equiv A' A_1$ so that the above state is a particular n^{th} extension of the state in (8). Bob receives the systems A' and M. The most general decoding operation that Bob can perform on his three registers A', B^n, and M is a conditional quantum decoder $\mathcal{D}^{M A' B^n \rightarrow B_1}$ consisting of a collection $\{\mathcal{D}_m^{A' B^n \rightarrow B_1}\}_m$ of CPTP maps. The state of Bob's system after the conditional quantum decoder $\mathcal{D}^{M A' B^n \rightarrow B_1}$ (at time t_f in Fig. 9) is as follows:

$$(\omega')^{A_1 E' B_1 E^n} \equiv \mathcal{D}^{M A' B^n \rightarrow B_1} (\omega^{A' A_1 M E' B^n E^n}).$$

Suppose that an $(n, C+\delta, Q+\delta, E-\delta, \epsilon)$ classically assisted state redistribution protocol as given above exists. We prove that the following bounds apply to the elements of its rate triple $(C+\delta, Q+\delta, E-\delta)$:

$$C + 2Q + \delta \geq \frac{1}{n} \left(I(M; E^n | B^n)_\omega + I(A' A_1; E^n | E' M)_\omega \right) \tag{85}$$

$$E - \delta \leq C + Q + \frac{1}{n} \left(I(A' A_1 \rangle B^n M)_\omega - I(M; E^n | B^n)_\omega \right) \tag{86}$$

$$E - \delta \leq \frac{1}{n} I(A' A_1 \rangle B^n M)_\omega + Q \tag{87}$$

for any $\epsilon, \delta > 0$ and all sufficiently large n. We redefine the system $A' \equiv A' A_1$ so that our expression above matches that in the direct-static capacity theorem.

In the ideal case, the protocol produces the maximally entangled state $\Phi^{A_1 B_1}$. So for our case, the following inequality

$$\left\| (\omega')^{A_1 B_1} - \Phi^{A_1 B_1} \right\|_1 \leq \epsilon \tag{88}$$

holds because the protocol is ϵ-good for entanglement generation.

We first prove the bound in (87)

$$\begin{aligned} n(E-\delta) &= I(A_1\rangle B_1)_{\Phi^{A_1B_1}} \\ &\le I(A_1\rangle B_1)_{\omega'} + n\delta' \\ &\le I(A_1\rangle A'B^nM)_\omega + n\delta' \\ &\le I(A_1A'\rangle B^nM)_\omega + H(A'|M)_\omega + n\delta' \\ &\le I(A_1A'\rangle B^nM)_\omega + nQ + n(\delta'+\delta). \end{aligned}$$

The first equality follows from evaluating the coherent information of the maximally entangled state $\Phi^{A_1B_1}$, the first inequality follows from the Alicki–Fannes' inequality [61] where we define $\delta' \equiv 4E\epsilon + H(\epsilon)/n$, the second inequality follows from quantum data processing [62], the third inequality follows because conditioning reduces entropy $(H(A'|B^nM)_\omega \le H(A'|M)_\omega)$, and the last inequality follows because $H(A'|M)_\omega \le H(A')_\omega \le n(Q+\delta)$ (the entropy of system A' has to be less than the log of the dimension of the system).

We next prove the lower bound in (85) on the classical and quantum communication consumption rate

$$\begin{aligned} &n(C+2Q+\delta) \\ &\quad \ge H(M)_\omega + Q + E - I(A'A_1\rangle B^nM)_\omega \\ &\quad \ge H(M|B^n)_\omega + H(A'|M)_\omega + H(A_1|M)_\omega \\ &\qquad - H(B^n|M)_\omega + H(A'A_1B^n|M)_\omega \\ &\quad \ge H(M|B^n)_\omega - H(M|B^nE^n)_\omega + H(A'A_1|M)_\omega \\ &\qquad - H(A'A_1E^nE'|M)_\omega + H(E^nE'|M)_\omega \\ &\quad = I(M;E^n|B^n)_\omega + I(A'A_1;E^nE'|M)_\omega \\ &\quad \ge I(M;E^n|B^n)_\omega + I(A'A_1;E^n|E'M)_\omega. \end{aligned}$$

The first inequality follows because the entropy of M is less than that of the uniform distribution and by exploiting the inequality in (87). The second inequality follows because conditioning reduces entropy $(H(M) \ge H(M|B^n))$, because $Q \ge H(A'|M)_\omega$ and $E \ge H(A_1|M)_\omega$, and by expanding the coherent information $I(A'A_1\rangle B^nM)_\omega$. The third inequality follows because $H(M|B^nE^n)_\omega \ge 0$, from subadditivity, and because the state on $B^nA'A_1E^nE'$ is pure when conditioned on the classical variable M. The sole equality follows by collecting terms, and the last inequality follows because $I(A'A_1;E^nE'|M)_\omega = I(A'A_1;E'|M)_\omega + I(A'A_1;E^n|E'M)_\omega I(A'A_1;E'|M)_\omega \ge 0$.

The inequality in (86) follows by exploiting (87) and that $nC \ge H(M)_\omega \ge I(M;E^n|B^n)_\omega$.

Appendix II
$(-,+,+)$ Octant of the Direct-Static Capacity Region

The technique for handling this octant is similar to the technique for handling the octant $(+,-,+)$. We give the full proof for completeness. Define

$$\mathcal{C}_{\text{DS}}^{-++}(\rho) \equiv \mathcal{C}_{\text{DS}}(\rho) \cap O^{-++}$$

and recall the definition of $\mathcal{C}_{\text{DS}}^{-+0}(\rho)$ in (37). Recall the line of TP L_{TP} as defined in (15).

Define the following maps:

$$\begin{aligned} f &: S \to (S + L_{\text{TP}}) \cap Q^{-+0} \\ \hat{f} &: S \to (S - L_{\text{TP}}) \cap O^{-++}. \end{aligned}$$

The map f translates the set S in the TP direction and keeps the points in the Q^{-+0} quadrant. The map $\hat{f}$, in a sense, undoes the effect of f by moving the set S back to the octant O^{-++}.

The inclusion $\mathcal{C}_{\text{DS}}^{-++}(\rho) \subseteq \hat{f}(f(\mathcal{C}_{\text{DS}}^{-++}(\rho)))$ holds because

$$\begin{aligned} \mathcal{C}_{\text{DS}}^{-++}(\rho) &= \mathcal{C}_{\text{DS}}^{-++}(\rho) \cap O^{-++} \\ &\subseteq (((\mathcal{C}_{\text{DS}}^{-++}(\rho) + L_{\text{TP}}) \cap Q^{-+0}) - L_{\text{TP}}) \cap O^{-++} \\ &= (f(\mathcal{C}_{\text{DS}}^{-++}(\rho)) - L_{\text{TP}}) \cap O^{-++} \\ &= \hat{f}(f(\mathcal{C}_{\text{DS}}^{-++}(\rho))). \end{aligned} \tag{89}$$

The first set equivalence is obvious from the definition of $\mathcal{C}_{\text{DS}}^{-++}(\rho)$. The first inclusion follows from the following logic. Pick any point $a \equiv (C,Q,E) \in \mathcal{C}_{\text{DS}}^{-++}(\rho) \cap O^{-++}$ and a particular point $b \equiv (-2E, E, -E) \in L_{\text{TP}}$. It follows that the point $a + b \in (\mathcal{C}_{\text{DS}}^{-++}(\rho) + L_{\text{TP}}) \cap Q^{-+0}$. We then pick a point $-b \equiv (2E, -E, E) \in -L_{\text{TP}}$. It follows that $a+b-b \in (((\mathcal{C}_{\text{DS}}^{-++}(\rho) + L_{\text{TP}}) \cap Q^{-+0}) - L_{\text{TP}}) \cap O^{-++}$ and that $a + b - b = (C,Q,E) = a$. The first inclusion thus holds because every point in $\mathcal{C}_{\text{DS}}^{-++}(\rho) \cap O^{-++}$ is in $(((\mathcal{C}_{\text{DS}}^{-++}(\rho) + L_{\text{TP}}) \cap Q^{-+0}) - L_{\text{TP}}) \cap O^{-++}$. The second set equivalence follows from the definition of f and the third set equivalence follows from the definition of $\hat{f}$.

It is operationally clear that the following inclusion holds:

$$f(\mathcal{C}_{\text{DS}}^{-++}(\rho)) \subseteq \mathcal{C}_{\text{DS}}^{-+0}(\rho) \tag{90}$$

because the mapping f converts any achievable point $a \in \mathcal{C}_{\text{DS}}^{-++}(\rho)$ to an achievable point in $\mathcal{C}_{\text{DS}}^{-+0}(\rho)$ by consuming all of the entanglement in a with TP.

The converse proof of classically assisted quantum communication [25] is useful for us

$$\mathcal{C}_{\text{DS}}^{-+0}(\rho) \subseteq \tilde{\mathcal{C}}_{\text{DS}}^{-+0}(\rho). \tag{91}$$

The inclusion $\hat{f}(\tilde{\mathcal{C}}_{\text{DS}}^{-+0}(\rho)) \subseteq \tilde{\mathcal{C}}_{\text{DS}}^{-++}(\rho)$ holds because

$$\begin{aligned} &\hat{f}(\tilde{\mathcal{C}}_{\text{DS}}^{-+0}(\rho)) \\ &\quad = (((\tilde{\mathcal{C}}_{\text{CASR}}(\rho) + L_{\text{TP}}) \cap Q^{-+0}) - L_{\text{TP}}) \cap O^{-++} \\ &\quad \subseteq ((\tilde{\mathcal{C}}_{\text{CASR}}(\rho) + L_{\text{TP}}) - L_{\text{TP}}) \cap O^{-++} \\ &\quad = ((\tilde{\mathcal{C}}_{\text{CASR}}(\rho) + L_{\text{TP}}) \cap O^{-++}) \\ &\qquad \cup ((\tilde{\mathcal{C}}_{\text{CASR}}(\rho) - L_{\text{TP}}) \cap O^{-++}) \\ &\quad \subseteq \tilde{\mathcal{C}}_{\text{DS}}^{-++}(\rho). \end{aligned} \tag{92}$$

The first set equivalence follows by definition. The first inclusion follows by dropping the intersection with Q^{-+0}. The second set equivalence follows because $(\tilde{\mathcal{C}}_{\text{CASR}}(\rho) + L_{\text{TP}}) - L_{\text{TP}} = (\tilde{\mathcal{C}}_{\text{CASR}}(\rho) + L_{\text{TP}}) \cup (\tilde{\mathcal{C}}_{\text{CASR}}(\rho) - L_{\text{TP}})$, and the last inclusion follows because $(\tilde{\mathcal{C}}_{\text{CASR}}(\rho) - L_{\text{TP}}) \cap O^{-++} = (0,0,0)$.

Putting (89)–(92) together, we have the following inclusion:

$$\begin{aligned}\mathcal{C}_{\mathrm{DS}}^{-++}(\rho) &\subseteq \hat{f}(f(\mathcal{C}_{\mathrm{DS}}^{-++}(\rho))) \\ &\subseteq \hat{f}(\mathcal{C}_{\mathrm{DS}}^{-+0}(\rho)) \subseteq \hat{f}(\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho)) \\ &\subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}^{-++}(\rho).\end{aligned}$$

The above inclusion $\mathcal{C}_{\mathrm{DS}}^{-++} \subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}^{-++}$ is the statement of the converse theorem for this octant.

Appendix III
$(-,+,-)$ Octant of the Direct-Static Capacity Region

The proof of this octant is similar to the proof of the octant $(+,-,-)$. We first need the following additivity lemma.

Lemma 5: The following inclusion holds:

$$\mathcal{C}_{\mathrm{DS}}^{-+0}(\rho\otimes\Phi^{|E|}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\Phi^{|E|}).$$

Proof: TP induces a linear bijection $f : \tilde{\mathcal{C}}_{\mathrm{DS}}^{-0+}(\rho) \rightarrow \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho)$ between the entanglement distillation achievable region $\tilde{\mathcal{C}}_{\mathrm{DS}}^{-0+}(\rho)$ and the noisy TP achievable region $\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho)$ [25]. The bijection f behaves as follows for every point $(C,0,E) \in \tilde{\mathcal{C}}_{\mathrm{DS}}^{-0+}(\rho)$:

$$f : (C,0,E) \rightarrow (C-2E, E, 0).$$

The following relation holds:

$$f(\tilde{\mathcal{C}}_{\mathrm{DS}}^{-0+}(\rho)) = \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho) \tag{93}$$

because applying TP to entanglement distillation gives noisy TP [25]. The inclusion $\mathcal{C}_{\mathrm{DS}}^{-0+}(\rho\otimes\Phi^{|E|}) \subseteq f^{-1}(\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\Phi^{|E|}))$ holds because

$$\begin{aligned}\mathcal{C}_{\mathrm{DS}}^{-0+}(\rho\otimes\Phi^{|E|}) &= \mathcal{C}_{\mathrm{DS}}^{-0+}(\rho) + (0,0,E) \\ &\subseteq \mathcal{C}_{\mathrm{DS}}^{-0+}(\rho) + \mathcal{C}_{\mathrm{DS}}^{-0+}(\Phi^{|E|}) \\ &= \tilde{\mathcal{C}}_{\mathrm{DS}}^{-0+}(\rho) + \tilde{\mathcal{C}}_{\mathrm{DS}}^{-0+}(\Phi^{|E|}) \\ &= f^{-1}(\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho)) + f^{-1}(\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\Phi^{|E|})) \\ &= f^{-1}(\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\Phi^{|E|})).\end{aligned}$$

The first set equivalence follows because the capacity region of the noisy resource state ρ combined with a rate E maximally entangled state is equivalent to a translation of the capacity region of the noisy resource state ρ. The first inclusion follows because the capacity region of a rate E maximally entangled state contains the rate triple $(0,0,E)$. The second set equivalence follows from (42), the third set equivalence from (93), and the fourth set equivalence from linearity of the map f. The above inclusion implies the following one:

$$f(\mathcal{C}_{\mathrm{DS}}^{-0+}(\rho\otimes\Phi^{|E|})) \subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\Phi^{|E|}).$$

The lemma follows because

$$\begin{aligned}f(\mathcal{C}_{\mathrm{DS}}^{-0+}(\rho\otimes\Phi^{|E|})) &= f(\tilde{\mathcal{C}}_{\mathrm{DS}}^{-0+}(\rho\otimes\Phi^{|E|})) \\ &= \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho\otimes\Phi^{|E|}) \\ &= \mathcal{C}_{\mathrm{DS}}^{-+0}(\rho\otimes\Phi^{|E|})\end{aligned}$$

where we apply the relations in (93) and (42). □

Observe that

$$\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\Phi^{|E|}) = \tilde{\mathcal{C}}_{\mathrm{U}}^{-+E}. \tag{94}$$

Hence, for all $E \leq 0$

$$\mathcal{C}_{\mathrm{DS}}^{-+E}(\rho) = \mathcal{C}_{\mathrm{DS}}^{-+0}(\rho\otimes\Phi^{|E|}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{U}}^{-+E} \tag{95}$$

where we apply Lemma 5 and (94). Thus

$$\begin{aligned}\mathcal{C}_{\mathrm{DS}}^{-+-}(\rho) &= \bigcup_{E\leq 0} \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+E}(\rho) \\ &\subseteq \bigcup_{E\leq 0} \left(\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{U}}^{-+E}\right) \\ &= (\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho) + \tilde{\mathcal{C}}_{\mathrm{U}}) \cap O^{-+-} \\ &\subseteq (\tilde{\mathcal{C}}_{\mathrm{CASR}}(\rho) + \tilde{\mathcal{C}}_{\mathrm{U}}) \cap O^{-+-} \\ &= \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+-}(\rho).\end{aligned}$$

The first set equivalence holds by definition. The first inclusion follows from (95). The second set equivalence follows because $\bigcup_{E\leq 0} \tilde{\mathcal{C}}_{\mathrm{U}}^{-+E} = \tilde{\mathcal{C}}_{\mathrm{U}} \cap O^{-+-}$. The second inclusion holds because $\tilde{\mathcal{C}}_{\mathrm{DS}}^{-+0}(\rho)$ is equivalent to noisy TP and the classically assisted state redistribution combined with the unit resource region generates noisy TP. The above inclusion $\mathcal{C}_{\mathrm{DS}}^{-+-}(\rho) \subseteq \tilde{\mathcal{C}}_{\mathrm{DS}}^{-+-}(\rho)$ is the statement of the converse theorem for this octant.

Appendix IV
$(-,+,+)$ Octant of the Direct-Dynamic Capacity Region

The proof of this octant is similar to the proof of the octant $(+,+,+)$. Define

$$\begin{aligned}\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) &\equiv \mathcal{C}_{\mathrm{DD}}(\mathcal{N}) \cap O^{-++} \\ \mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N}) &\equiv \mathcal{C}_{\mathrm{DD}}(\mathcal{N}) \cap Q^{-0+}.\end{aligned}$$

Recall the definition of the line of ED L_{ED} in (17). Define the following maps:

$$\begin{aligned}f &: S \rightarrow (S + L_{\mathrm{ED}}) \cap Q^{-0+} \\ \hat{f} &: S \rightarrow (S - L_{\mathrm{ED}}) \cap O^{-++}.\end{aligned}$$

The map f translates the set S in the ED direction and keeps the points that lie on the Q^{-0+} quadrant. The map $\hat{f}$, in a sense, undoes the effect of f by moving the set S back to the O^{-++} octant.

The inclusion $\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) \subseteq \hat{f}(f(\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N})))$ holds because

$$\begin{aligned}&\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) \\ &\quad= \mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) \cap O^{-++} \\ &\quad\subseteq (((\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{-0+}) - L_{\mathrm{ED}}) \cap O^{-++} \\ &\quad= (f(\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N})) - L_{\mathrm{ED}}) \cap O^{-++} \\ &\quad= \hat{f}(f(\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}))).\end{aligned} \tag{96}$$

The first set equivalence is obvious from the definition of $\mathcal{C}_{\mathrm{DD}}^{-++}$. The first inclusion follows from the following logic. Pick any point $a \equiv (C,Q,E) \in \mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) \cap O^{-++}$ and a particular point $b \equiv (0, -Q, Q) \in L_{\mathrm{ED}}$. It follows that the point $a + b = (C, 0, E + Q) \in (\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{-0+}$. We then pick a point $-b = (0, Q, -Q) \in -L_{\mathrm{ED}}$. It follows that $a + b - b \in (((\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{-0+}) - L_{\mathrm{ED}}) \cap O^{-++}$ and that $a + b - b = (C, Q, E) = a$. Thus, the first inclusion follows because every point in $\mathcal{C}_{\mathrm{DD}}^{-++} \cap O^{-++}$ belongs to $(((\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) + L_{\mathrm{ED}}) \cap Q^{-0+}) - L_{\mathrm{ED}}) \cap O^{-++}$. The second set equivalence follows from the definition of f, and the third set equivalence follows from the definition of $\hat{f}$.

It is operationally clear that the following inclusion holds:

$$f(\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N})) \subseteq \mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N}) \tag{97}$$

because the mapping f converts any achievable point in $\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N})$ to an achievable point in $\mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N})$ by combining it with ED.

Forward classical communication does not increase the entanglement generation capacity [16], [34]. Thus, the following result from (68) applies:

$$\mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N}) = \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}). \tag{98}$$

It then follows that

$$\begin{aligned}\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) &= \tilde{\mathcal{C}}_{\mathrm{DD}}^{00+}(\mathcal{N}) + L^{-00}\\ &= (((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) \cap Q^{0+-}) + L_{\mathrm{ED}}) \cap L^{00+}) + L^{-00}\\ &\subseteq \tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{ED}} + L^{-00}\\ &= \tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{ED}} + L_{\mathrm{TP}}.\end{aligned} \tag{99}$$

The first set equivalence follows from (68). The second set equivalence follows from (69). The first inclusion follows by dropping the intersections with Q^{0+-} and L^{00+}, and the last inclusion follows because ED and TP can generate any point along L^{-00}.

The inclusion $\hat{f}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N})) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{-++}(\mathcal{N})$ holds because

$$\begin{aligned}&\hat{f}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}))\\ &\quad\subseteq ((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{TP}} + L_{\mathrm{ED}}) - L_{\mathrm{ED}}) \cap O^{-++}\\ &\quad= ((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{TP}} + L_{\mathrm{ED}}) \cap O^{-++})\\ &\quad\quad\cup ((\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{TP}} - L_{\mathrm{ED}}) \cap O^{-++})\\ &\quad\subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{-++}(\mathcal{N}).\end{aligned} \tag{100}$$

The first inclusion follows from (99) and the definition of $\hat{f}$. The first set equivalence follows because $(\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{TP}} + L_{\mathrm{ED}}) - L_{\mathrm{ED}} = (\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{TP}} + L_{\mathrm{ED}}) \cup (\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{TP}} - L_{\mathrm{ED}})$, and the last inclusion follows because $(\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + L_{\mathrm{TP}} - L_{\mathrm{ED}}) \cap O^{-++} = (0, 0, 0)$ and (14).

Putting (96)–(98) and (100) together, the following inclusion holds:

$$\begin{aligned}\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) &\subseteq \hat{f}(f(\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N})))\\ &\subseteq \hat{f}(\mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N})) \subseteq \hat{f}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N})) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{-++}(\mathcal{N}).\end{aligned}$$

The above inclusion $\mathcal{C}_{\mathrm{DD}}^{-++}(\mathcal{N}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{-++}(\mathcal{N})$ is the statement of the converse theorem for this octant.

Appendix V
$(-,-,+)$ Octant of the Direct-Dynamic Capacity Region

The proof technique for this octant is similar to that for the $(+,-,-)$ octant in the static case. We exploit the bijection between the quantum communication achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DD}}^{0+0}$ and the entanglement generation achievable rate region $\tilde{\mathcal{C}}_{\mathrm{DD}}^{00+}$. We need the following lemma.

Lemma 6: The following inclusion holds:

$$\mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathrm{id}^{\otimes|Q|}).$$

Proof: ED induces a bijective mapping $f : \mathcal{C}_{\mathrm{DD}}^{-+0}(\mathcal{N}) \to \mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N})$ between the classically assisted quantum communication achievable region and the classically assisted entanglement generation achievable region. It behaves as follows for every point $(C, Q, 0) \in \mathcal{C}_{\mathrm{DD}}^{-+0}(\mathcal{N})$:

$$f : (C, Q, 0) \to (C, 0, Q).$$

The following relation holds:

$$f(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-+0}(\mathcal{N})) = \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) \tag{101}$$

because applying ED to the classically assisted quantum communication protocol produces classically assisted entanglement generation. The inclusion $\mathcal{C}_{\mathrm{DD}}^{-+0}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|}) \subseteq f^{-1}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathrm{id}^{\otimes|Q|}))$ holds because

$$\begin{aligned}&\mathcal{C}_{\mathrm{DD}}^{-+0}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|})\\ &\quad= \mathcal{C}_{\mathrm{DD}}^{-+0}(\mathcal{N}) + (0, Q, 0)\\ &\quad\subseteq \mathcal{C}_{\mathrm{DD}}^{-+0}(\mathcal{N}) + \mathcal{C}_{\mathrm{DD}}^{-+0}(\mathrm{id}^{\otimes|Q|})\\ &\quad= \tilde{\mathcal{C}}_{\mathrm{DD}}^{-+0}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{DD}}^{-+0}(\mathrm{id}^{\otimes|Q|})\\ &\quad= f^{-1}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N})) + f^{-1}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathrm{id}^{\otimes|Q|}))\\ &\quad= f^{-1}(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathrm{id}^{\otimes|Q|})).\end{aligned}$$

The first set equivalence follows because the capacity region of the noisy channel $\mathcal{N}$ combined with a rate Q noiseless qubit channel is equivalent to a translation of the capacity region of the noisy channel $\mathcal{N}$. The first inclusion follows because the capacity region of a rate Q noiseless qubit channel contains the rate triple $(0, Q, 0)$. The second set equivalence follows from the classically assisted quantum communication theorem in (68), the third set equivalence from (101), and the fourth set equivalence from linearity of the map f. The above inclusion implies the following one:

$$f(\mathcal{C}_{\mathrm{DD}}^{-+0}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|})) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathrm{id}^{\otimes|Q|}).$$

The lemma follows because

$$\begin{aligned}f(\mathcal{C}_{\mathrm{DD}}^{-+0}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|})) &= f(\tilde{\mathcal{C}}_{\mathrm{DD}}^{-+0}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|}))\\ &= \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|})\\ &= \mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|})\end{aligned}$$

where we apply the relations in (68), (101), and (70). □

Observe that

$$\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathrm{id}^{\otimes|Q|}) = \tilde{\mathcal{C}}_{\mathrm{U}}^{-Q+}. \tag{102}$$

Hence, for all $Q \leq 0$

$$\mathcal{C}_{\mathrm{DD}}^{-Q+}(\mathcal{N}) = \mathcal{C}_{\mathrm{DD}}^{-0+}(\mathcal{N} \otimes \mathrm{id}^{\otimes|Q|}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{U}}^{-Q+} \tag{103}$$

where we apply Lemma 6 and (102). The inclusion $\mathcal{C}_{\mathrm{DD}}^{--+}(\mathcal{N}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{--+}(\mathcal{N})$ holds because

$$\begin{aligned}\mathcal{C}_{\mathrm{DD}}^{--+}(\mathcal{N}) &= \bigcup_{Q\leq 0} \mathcal{C}_{\mathrm{DD}}^{-Q+}(\mathcal{N}) \\ &\subseteq \bigcup_{Q\leq 0} (\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{U}}^{-Q+}) \\ &= (\tilde{\mathcal{C}}_{\mathrm{DD}}^{-0+}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{U}}) \cap O^{--+} \\ &\subseteq (\tilde{\mathcal{C}}_{\mathrm{CEF}}(\mathcal{N}) + \tilde{\mathcal{C}}_{\mathrm{U}}) \cap O^{--+} \\ &= \tilde{\mathcal{C}}_{\mathrm{DD}}^{--+}(\mathcal{N}).\end{aligned}$$

The first set equivalence holds by definition. The first inclusion follows from (103). The second set equivalence follows because $\bigcup_{Q\leq 0} \tilde{\mathcal{C}}_{\mathrm{U}}^{-Q+} = \tilde{\mathcal{C}}_{\mathrm{U}} \cap O^{--+}$. The second inclusion follows because combining the classically enhanced father region with ED and TP gives the region for classically assisted entanglement generation. The above inclusion $\mathcal{C}_{\mathrm{DD}}^{--+}(\mathcal{N}) \subseteq \tilde{\mathcal{C}}_{\mathrm{DD}}^{--+}(\mathcal{N})$ is the statement of the converse theorem for this octant.

Appendix VI
Information-Theoretic Argument for the Converse of the $(-,+,-)$ Dynamic Octant

We provide an information-theoretic proof of the following bounds for all rate triples $(-|C|, Q, -|E|)$ in the $(-,+,-)$ dynamic octant (classical- and entanglement-assisted quantum communication):

$$2nQ \leq I(AX;B^n) + n|C| \tag{104}$$
$$nQ \leq I(A\rangle B^n X) + n|E|. \tag{105}$$

Fig. 10 depicts the most general protocol for classical- and entanglement-assisted quantum communication. Alice wishes to transmit the A_1 system of a maximally entangled state Φ^{RA_1} and shares a maximally entangled state $\Phi^{T_A T_B}$ with Bob on systems T_A and T_B. Her initial state is as follows:

$$\omega^{RA_1T_AT_B} \equiv \Phi^{RA_1} \otimes \Phi^{T_AT_B}.$$

She performs a quantum instrument $T^{A_1T_A \longrightarrow A'^nM}$ on systems A_1 and T_A to produce a quantum system A'^n and a classical system M where A'^n goes to the noisy quantum channel and M goes to the noiseless classical channel. The state is then

$$\omega^{RA'^nMT_B} \equiv T^{A_1T_A \longrightarrow A'^nM}(\omega^{RA_1T_AT_B}).$$

The channel $\mathcal{N}$ transforms A'^n to B^n to produce the following state:

$$\omega^{RB^nMT_B} \equiv \mathcal{N}^{A'^n \longrightarrow B^n}(\omega^{RA'^nMT_B}). \tag{106}$$

At this point, the state is almost a state of the form in (13) with $A \equiv RT_B$ (more on this later). Bob combines the classical

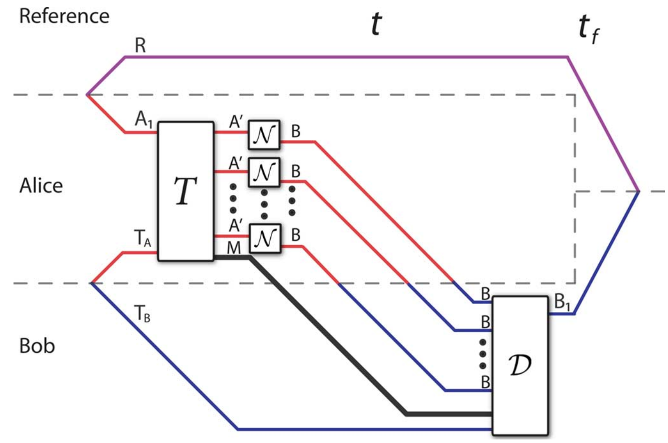


Fig. 10. Most general protocol for quantum communication with the help of a noisy channel, noiseless entanglement, and noiseless classical communication. Alice wishes to communicate a quantum register A_1 to Bob. She shares entanglement with Bob in the form of maximally entangled states. Her half of the entanglement is in system T_A and Bob's half is in the system T_B. Alice performs some quantum instrument T on her quantum register and her half of the entanglement. The output of this instrument is a classical message in some register M and a large number of systems A'^n that are input to the channel. She transmits A'^n through the noisy channel and the system M over noiseless classical channels. Bob receives the outputs B^n of the channel and the register M from the noiseless classical channels. He combines these with his half of the entanglement and decodes the quantum state that Alice transmits.

system M and the quantum systems B^n and T_B at a conditional quantum channel $\mathcal{D}^{MB^nT_B \longrightarrow B_1}$ to produce the state B_1 giving the following state:

$$(\omega')^{RB_1} \equiv \mathcal{D}^{MB^nT_B \longrightarrow B_1}(\omega^{RB^nMT_B}).$$

The protocol is ϵ-good if the state ω' is close in trace distance to Alice's original state Φ^{RA_1}

$$\left\|(\omega')^{RB_1} - \Phi^{RA_1}\right\|_1 \leq \epsilon. \tag{107}$$

We first prove the bound in (104). Consider the following chain of inequalities:

$$\begin{aligned} n2Q &= I(R;B_1)_\Phi \\ &\leq I(R;B_1)_{\omega'} + n\delta' \\ &\leq I(R;B^nT_BM)_\omega + n\delta' \\ &= I(R;B^n|T_BM)_\omega + I(R;T_BM)_\omega + n\delta' \\ &= I(RT_BM;B^n)_\omega - I(T_BM;B^n)_\omega \\ &\quad + I(R;T_BM)_\omega + n\delta' \\ &\leq I(RT_BM;B^n)_\omega + I(R;T_B)_\omega \\ &\quad + I(R;M|T_B)_\omega + n\delta' \\ &= I(RT_BM;B^n)_\omega + H(M|T_B)_\omega \\ &\quad - H(M|T_BR)_\omega + n\delta' \\ &\leq I(AM;B^n)_\omega + n|C| + n\delta'. \end{aligned}$$

The first equality follows by evaluating the quantum mutual information on the maximally entangled state Φ^{RB_1}. The first inequality follows from the condition in (107) and from a variation of the Alicki–Fannes' inequality with $\delta' \equiv 5|Q|\epsilon + 3H_2(\epsilon)/n$ [31, Corollary 1]. The second inequality follows from the quantum data processing inequality [35]. The third and fourth equalities follow by expanding the quantum mutual information $I\left(R;B^{n}T_{B}M\right)_{\omega}$ with the chain rule. The third inequality follows because $I\left(T_{B}M;B^{n}\right)_{\omega}\geq 0$ and by expanding the quantum mutual information $I\left(R;T_{B}M\right)_{\omega}$ with the chain rule. The fourth equality follows because $I\left(R;T_{B}\right)_{\omega}=0$ for this protocol and by rewriting the mutual information $I\left(R;M|T_{B}\right)_{\omega}$. The last inequality follows because $H\left(M|T_{B}\right)_{\omega}\leq n\left|C\right|$ and $I\left(A_{1};T_{B}|M\right)_{\omega}=H\left(A_{1}|M\right)_{\omega}-H\left(A_{1}|T_{B}M\right)_{\omega}$. The final inequality follows from the definition $A\equiv A_{1}T_{B}$ and because $H\left(M|T_{B}R\right)_{\omega}\geq 0$.

We now prove the bound in (104). Consider the following chain of inequalities:

$$\begin{aligned}
nQ &= I\left(R\rangle B_{1}\right)_{\Phi}\\
&\leq I\left(R\rangle B_{1}\right)_{\omega'}+n\delta'\\
&\leq I\left(R\rangle B^{n}T_{B}M\right)_{\omega}+n\delta'\\
&= H\left(B^{n}T_{B}M\right)_{\omega}-H\left(RB^{n}T_{B}M\right)_{\omega}+n\delta'\\
&= H\left(B^{n}M\right)_{\omega}+H\left(T_{B}|B^{n}M\right)_{\omega}\\
&\quad -H\left(RB^{n}T_{B}M\right)_{\omega}+n\delta'\\
&\leq I\left(A\rangle B^{n}M\right)_{\omega}+n\left|E\right|+n\delta'.
\end{aligned}$$

The first equality follows by evaluating the coherent information of the maximally entangled state $\Phi^{RB_{1}}$. The first inequality follows from the condition in (107) and from the Alicki–Fannes' inequality with $\delta'\equiv 4\left|Q\right|\epsilon+2H_{2}\left(\epsilon\right)/n$. The third inequality follows from quantum data processing [35]. The next two equalities follow by expanding the coherent information. The final inequality follows from the definition $A\equiv RT_{B}$ and because $H\left(T_{B}|B^{n}M\right)_{\omega}\leq n\left|E\right|$.

We should make some final statements concerning this proof. The state in (106) as we have defined it is not quite a state of the form in (13) because the instrument has an environment. Though, a few arguments demonstrate that a particular type of instrument works just as well as a general instrument, and it then follows that the state in (106) is of the form in (13). First, consider that a general instrument $T^{A_{1}T_{A}\rightarrow A'^{n}M}$ has a realization as an isometry $U_{T}^{A_{1}T_{A}\rightarrow A'^{n}E'E_{M}}$ followed by a von Neumann measurement of the system E_{M} in the basis $\left\{\left|m\right\rangle\left\langle m\right|^{M}\right\}$ (see the discussion of the CP formalism in [25]). The system E' is not involved in any of the entropic expressions in (104) and (105). Thus, Alice can measure the system E' in some classical basis $\left|l\right\rangle\left\langle l\right|$, obtaining a classical variable L, and a new state σ whose entropies in (104) and (105) are the same as those of the original state ω. Additionally, the action of Alice's von Neumann measurement of E' makes the state σ be a state of the form in (13). The following inequalities then hold by the quantum data processing inequality:

$$\begin{aligned}
I\left(AM;B^{n}\right)_{\omega} &\leq I\left(AML;B^{n}\right)_{\sigma}\\
I\left(A\rangle B^{n}M\right)_{\omega} &\leq I\left(A\rangle B^{n}ML\right)_{\sigma}
\end{aligned}$$

demonstrating that it is sufficient to consider states of the form in (13) for determining the capacity region for this octant.

ACKNOWLEDGMENT

The authors would like to thank I. Devetak, P. Hayden, and D. Leung for initial discussions during the development of this project, and the anonymous referee for bringing the classically assisted quantum state redistribution protocol to their attention. M. M. Wilde also acknowledges the hospitality of the ERATO-SORST project, where he and M.-H. Hsieh were able to complete this research.

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